DIFFERENTIAL AND INTEGRAL

CALCULUS

JAMES LEE LOVE, A. M.

To Prof. W. F. Osgood
with cordial regards of
James Lee Love.

AN INTRODUCTORY COURSE

IN THE

DIFFERENTIAL AND INTEGRAL CALCULUS

FOR STUDENTS IN ENGINEERING IN THE LAWRENCE SCIENTIFIC SCHOOL

BY

JAMES LEE LOVE, A.M.
INSTRUCTOR IN MATHEMATICS IN HARVARD UNIVERSITY

CAMBRIDGE
Published by Harvard University
1899

PREFACE.

This Course in the Differential and Integral Calculus has been prepared solely for students in the Lawrence Scientific School of Harvard University. These students are young men who are pursuing the study chiefly for the sake of its applications in Engineering and kindred branches; who wish, therefore, to gain facility in the use of the Calculus as a tool of investigation. In order to realize this end, the essentials of the subject should be presented in such form as to secure, by sound reasoning and clear demonstration, the mental discipline characteristic of the study; in such notation as the student is likely to meet in subsequent reading; and in such relations to its applications as to promote interest in the work.

But the student should remember that, here, *the applications are subordinate; and that his main purpose is to learn the principles and methods of the Calculus.* He cannot be expected to apply the Calculus, except in a very limited way, to questions in Mechanics, Hydraulics, Resistance of Materials, Bridges, Steam Engines, Electric Machinery, etc., until he has learned, first, the formulæ of the Calculus; and, secondly, the rudiments, at least, of the study in which he wishes to use his presumed mathematical knowledge.

Perhaps the only noteworthy departure from the conventional presentation of the elements of the Calculus, aside from personal peculiarities of arrangement and statement, will be found in the attempt to avoid the common double use of the term "Integration": that is, as signifying both, "The Inverse of Differentiation"; and,

"A Process of Summation." Much confusion for the student is caused by this twofold meaning of the word: and the term "Anti-differentiation" is employed herein for "The Inverse of Differentiation"; while the term "Integration" is restricted to the distinctive Calculus "Process of Summation."

It would be a pleasing and becoming task to acknowledge my great indebtedness to many men and books in the preparation of this work, but it is quite impossible. To the admirable treatises of Professor William E. Byerly, and to his inspiring instruction, I am under special obligations.

I must thank my colleagues, Mr. A. B. Frizell and Dr. D. F. Campbell, for much kind assistance in revising and correcting the proofs.

In the preparation of this Course, the needs of my own classes alone have been before me; and a strong desire to help them has been my motive for the undertaking. If they find it a clear, interesting, and stimulating introduction to a great branch of Science, I shall feel abundantly satisfied.

JAMES LEE LOVE.

HARVARD UNIVERSITY,
May, 1899.

TABLE OF CONTENTS.

EMENDATIONS.

Page 14. *Put* Ex. 7 *after* Ex. 8.

Page 15, *title of* § 10. "Increment" *should be* "increments".

Page 25. *Omit* Exs. 6*a*, 6*b*, and 6*c*.

Page 27, *lines* 21, 22, *from top. Interchange* "increasing" *with* "decreasing".

Page 29, *line* 7 *from top. It is assumed that* $D_x^2 f(x)$ *is not zero at* x_1. *This case is treated in Exs.* 2, 3, p. 37.

Page 32, *in Probl.* 1. *Insert*, "with square base", *after* "box".

Page 56, *in* (6). *Read* "$\int e^{ax}\sin x\,dx$" *instead of*, "$\int e^{ax}\sin x$".

Page 83, *in Ex.* 1*a*. *The result*, "$\lim_{x=\infty} x e^{-ax} = 0$", *is required in solving* 1*a*.

Page 91. *Use* "E", *instead of* "F", *to denote the volume generated.*

Page 94, *line* 6 *from foot. Insert*, "and finite", *after* "single-valued".

Page 96, (4). *Change* "$+$" to "$-$", *before* $f(x_1)$, *in the* 2d *numerator.*

Page 97, *line* 2 *from top. Insert*, "is finite and", *before*, "has a derivative".

Page 101, *third line from top. Insert*, "$\infty - \infty$", *after*, "$0 \times \infty$".

Page 111, *twelfth line from top. Read* "convergent", *instead of* "convergcnt".

Page 112, *last line. Read* "$\frac{a_2}{3}x^3$", *instead of* "$\frac{a^2}{3}x^3$".

Page 113, (5). *Read* "a_3x^3", *instead of* "a_3a^3".

Page 124, *Ex.* 2*c*. *Change lower limit from* "0" *to* "$x_1 > 1$", *and add* k *to right member.*

Page 125, *last line of foot-note. Read* "$f''(0)$", *instead of* $f'(0)$.

Page 150, *Ex.* 4. *Read* "p^2", *instead of* "p^3" *in Ans.*

Page 153, (8). *Supply* "θ_0" *instead of* "0", *under* "Σ".

Page 191, *Ex.* 4. *The lower limit of integration is* "0".

Page 217, *line* 12 *from foot. Strike out* "(3)".

Page 220, *line* 3 *from top. Change period to semi-colon at end of line.*

DIFFERENTIAL CALCULUS.

CHAPTER I.

CONSTANTS, VARIABLES, FUNCTIONS AND FUNCTIONAL SYMBOLS.

1. CONSTANTS: VARIABLES.

A symbol representing a number which retains a fixed value in a given problem, is called a *constant.*

Constants are expressed either by *numerical symbols*, as 64, $\frac{35}{47}$, $\sqrt{14}$, $\sin 30°$, $\log 75$, etc.; or by *literal symbols*, as a, b, a_1, a_2, x_1, x_2, y_1, y_2, α_1, α_2, etc.

A *numerical constant* represents the same number everywhere; while a *literal constant* may represent different numbers in different problems. On this account, the latter are called *arbitrary constants.*

A symbol representing a number which is supposed to change value in some specified way, is called a *variable.*

Variables will be usually expressed by the symbols x, y, z, θ, ϕ, etc.

Such symbols as x_1, x_2, y_1, y_2, θ_1, θ_2, etc., will be used to express assigned, or arbitrarily fixed, values of the variable to which the subscript is affixed.

2. FUNCTIONS: EXPLICIT AND IMPLICIT.

The expression x^2, which will vary in value as x varies, and will take a determinate value when a fixed value has been assigned to x, is called a function of x.

Any such expression as $\sqrt[3]{x}$, $\sin x$, $\sin^{-1}x$, 4^x, $\log x$, or, in general, any combination of them, *will vary in value as x varies, will take one or more determinate values when a fixed value has been properly assigned to x, and, therefore, is called a function of x.*

A *function* is, therefore, a variable: but its variation in value depends upon the form of the function, as well as upon the manner in which we make its variable change value.

For this last reason any given function, as $x^2 + 3x + 4$, or $\sin x$, or $\log x$, is called a *dependent* variable; while its variable x is called an *independent* variable, because x is assumed to be capable of taking values assigned to it at our pleasure.

If the equation $x^2 + y^2 - a^2 = 0$ is given, it may be solved either for y or x; that is, we may obtain from it either $y = \pm\sqrt{[a^2 - x^2]}$, or $x = \pm\sqrt{[a^2 - y^2]}$. In the first result, $y = \pm\sqrt{[a^2 - x^2]}$, it is understood that x is the independent variable, and y the dependent variable, or function; and y is, in such a case, called an *explicit function of x*. In the second result, $x = \pm\sqrt{[a^2 - y^2]}$, x is an explicit function of the independent variable y.

When x and y are connected in an equation, as $x^2 + y^2 - a^2 = 0$, or $\sin(x + y) + e^x + \log y = 0$, which is not solved for either variable, the variable y is called an *implicit function of x if we wish to consider x as independent*, or x is called an *implicit function of y if y is to be considered as independent variable.*

From an equation involving x and y, we may obtain, either y as an explicit function of x by solving the equation for y, or x as an explicit function of y by solving for x. In many cases, however, as in $\sin(x + y) + e^x + \log y = 0$, neither solution may be practicable.

The functions, called "the ordinary functions," to be treated in this course, are classified as either (a) Algebraic, or (b) Transcendental. The Algebraic are those which involve *only the algebraic operations* (such as

$$4x^3 + 5x^2 - 8x + 3, \qquad \frac{3x^2 + 2x - 1}{5x^3 + 7},$$

$$\sqrt{[3x^2 + 2x + 1]}, \qquad x^{\frac{3}{4}} + 3, \text{ etc.}).$$

The transcendental include

(1) the Trigonometric functions, $\sin x$, $\cos x$, etc.;

(2) the anti-Trigonometric, $\sin^{-1}x$, $\cos^{-1}x$, etc.;

(3) the logarithmic, $\log x$, etc.;

(4) the anti-logarithmic or exponential, e^x, a^x, etc.

A function may be *mixed;* that is, composed of any combination of the algebraic and transcendental.

3. FUNCTIONAL SYMBOLS.

The symbol $f(x)$ is used as an arbitrary general symbol to denote any explicit function of x whatsover. It is read "f function of x"; or, more briefly, "f of x." The equation $y = f(x)$ means "y is some explicit function of x": it is read "y equals f of x." When a perfectly general theorem is being treated, one relating to a class of explicit functions, the symbol $f(x)$ will be used to represent them; and will be subject only to the limitations imposed by the theorem in question. Or, the symbol $f(x)$ may be used as a mere abbreviation for a function which reappears so often in the course of an investigation, as to make it desirable to have a short symbol to represent it. Instances of both these uses will be frequent.

Other symbols, used in precisely the same way as $f(x)$, when more than one function enters the same investigation, are $\phi(x)$, $\psi(x)$, $F(x)$, etc.

The single letters u, v, and w, will be used, as more abbreviated symbols, for the same purpose as $f(x)$, $\phi(x)$, etc.

The symbols $f(x, y) = 0$, $\phi(x, y) = 0$, etc., will be used to represent *implicit functions.*

Exercises.

1. Given $f(x) = x^3 - 4x$, show that $f(0) = 0$, $f(1) = -3$, $f(2) = 0$, $f(3) = 15$.

2. If $f(x) = \sin x$, find $f(0)$, $f\left(\frac{\pi}{6}\right)$, $f\left(\frac{\pi}{4}\right)$, $f\left(\frac{\pi}{3}\right)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$, $f\left(\frac{3\pi}{2}\right)$, $f(2\pi)$.

3. If $f(x) = \cos x$, find its value when x has each of the values given in ex. 2.

4. If $\phi(x) = a^x$, show that
$$\phi(2) \times \phi(3) = \phi(5); \quad \phi(y+z) = \phi(y) \times \phi(z).$$

5. If $F(x) = \cos x + \sqrt{-1}\sin x$, show that
$$F(2x) = [F(x)]^2; \quad F(x) = [F(\tfrac{1}{2}x)]^2.$$

CHAPTER II.

GEOMETRIC REPRESENTATION OF FUNCTIONS: INCREMENTS OF VARIABLE AND FUNCTION.

4. GEOMETRIC REPRESENTATION OF FUNCTIONS.

Any functional relation $f(x, y) = 0$, or $y = f(x)$, containing only two real variables, can generally be represented geometrically by means of rectangular axes, as explained in plane analytic geometry. For example, the function $y = \sin x$ will be represented, for values of x extending from $OA = -\pi$ to $OD = +\pi$, by the curve $ABOCD$. The ordinate, MP, of any point P of the curve, represents the value of the function y corresponding to the value, OM, of its variable x.

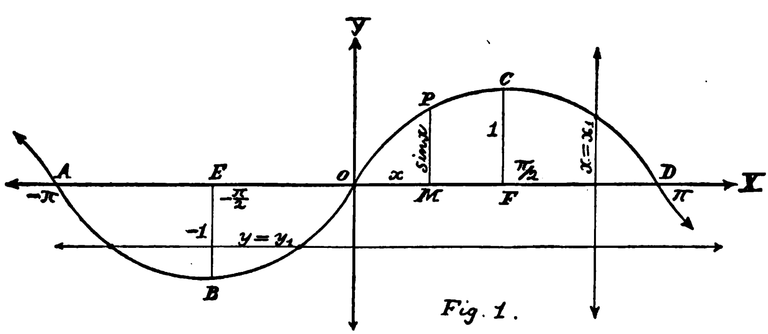

Fig. 1.

The curve representing a given function shows to the eye, immediately, relations between the function and its variable, — shows exactly how any specified change in the value of the one affects the value of the other. The curve $ABOCD$ in fig. 1, contains the entire table of natural sines of angles. The portion ABO, being below the x-axis, shows that $\sin x = y$ is negative if the angle x is between $-\pi$ and 0. The portion OCD, above the x-axis, shows that $\sin x = y$ is positive if the angle x is between 0 and π.

If we assume the variable x to take, successively, a set of values increasing from $OA = -\pi$ to $OE = -\frac{\pi}{2}$, the falling curve, from A to B, shows that the function $\sin x = y$ is decreasing. The rising curve, from B to C, shows that the function $\sin x = y$ is increasing, as the variable is made to increase from $OE = -\frac{\pi}{2}$ to $OF = \frac{\pi}{2}$, etc. If, moreover, the curve is accurately plotted, it may be used to furnish the approximate value of the sine ($= MP$) of any angle whose circular measure ($= OM$) is given. The curve, therefore, is as useful as a table of natural sines; and has the enormous advantage of giving the table in extremely condensed pictorial form.*

Equations containing only two variables can, generally, be represented geometrically in rectangular coördinates; but in some cases polar coördinates will be found most convenient. Any function will be regarded as completely represented by its geometric locus; and inferences drawn from a study of the curve, when rightly understood and interpreted, will furnish most important information concerning the function it represents; and this, in turn, will yield valuable knowledge of the mechanical, physical, or other, laws, which may be expressed in the function.

---

* Of course, the curve shown in fig. 1 is only a part of the whole curve representing $y = \sin x$. If x increases from $OD = \pi$ up to $x = 3\pi$, we shall get a repetition of the curve shown in the figure; another repetition will be obtained between $x = 3\pi$ and $x = 5\pi$; another, between $x = 5\pi$ and $x = 7\pi$; and so on, without stopping. Similarly, if x is made to decrease from $-\pi$ to -3π, then from -3π to -5π, — and so on, endlessly, to the left. The whole curve, representing the function $y = \sin x$, is, therefore, a wave curve extending along the x-axis, consisting of an endless number of repetitions of the part $ABOCD$.

It will be observed that every straight line parallel to OY, in fig. 1, cuts the curve representing $y = \sin x$, in one point only. This is the geometric expression of the fact, that, to any assigned real value of x, say $x = x_1$, will correspond one, and only one, value of the function $\sin x = y$.

Again, any line parallel to OX, provided its distance from OX is not greater than $FC = 1$, nor less than $EB = -1$, will cut the curve in many points. This is the geometric expression of the fact, that, if we regard y as independent variable, and change $y = \sin x$ to the equivalent form $x = \sin^{-1} y$, then, to any assigned value of y, $-1 \leqq y_1 \leqq 1$, will correspond many values of the function $x = \sin^{-1} y$.

Hence, in $y - \sin x = 0$, *y is a single-valued function of x, but x is a many-valued function of y.*

The following equations are represented by familiar curves:

$a)$ $y - mx - b = 0$ [a straight line whose slope $= m$];

$b)$ $x^2 + y^2 - a^2 = 0$ [a circle whose tangent at (x_1, y_1) is $x_1x + y_1y - a^2 = 0$];

$c)$ $b^2x^2 + a^2y^2 - a^2b^2 = 0$ [an ellipse whose tangent at (x_1, y_1) is $b^2x_1x + a^2y_1y - a^2b^2 = 0$];

$d)$ $b^2x^2 - a^2y^2 - a^2b^2 = 0$ [an hyperbola whose tangent at (x_1, y_1) is $b^2x_1x - a^2y_1y - a^2b^2 = 0$];

$e)$ $y^2 - 4px = 0$ [a parabola whose tangent at (x_1, y_1) is $y_1y - 2p(x + x_1) = 0$].

Exercises.

1. Using rectangular axes, and taking an inch as unit length, plot the curves representing the following functions:

$a)$ $y = x^{-2}$, $b)$ $y = x^{-1}$, $c)$ $y = x^{-\frac{1}{2}}$, $d)$ $y = x^{\frac{1}{2}}$, $e)$ $y = x$, $f)$ $y = x^2$, $g)$ $y = x^3$.

2. In the same way plot the curves

$a)$ $y = x + 1$, $b)$ $y = x - 1$, $c)$ $y = x^2 + 1$, $d)$ $y = x^2 - 1$, $e)$ $y = x + x^2$, $f)$ $y = x - x^2$.

5. INCREMENTS.

Let the function

$$(1) \qquad y = x^3 - 6x^2 + 9x$$

be given. Assign to x the set of values shown in the table below; and compute the corresponding values of the function y, as seen in the table.

$y = x^3 - 6x^2 + 9x$

$x =$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\Delta x =$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
$y =$	-16	$-6\frac{1}{8}$	0	$3\frac{1}{8}$	4	$3\frac{3}{8}$	2	$\frac{5}{8}$	0	$\frac{7}{8}$	4	$10\frac{1}{8}$	20
$\Delta y =$	$9\frac{7}{8}$	$6\frac{1}{8}$	$3\frac{1}{8}$	$\frac{7}{8}$	$-\frac{5}{8}$	$-1\frac{3}{8}$	$-1\frac{3}{8}$	$-\frac{5}{8}$	$\frac{7}{8}$	$3\frac{1}{8}$	$6\frac{1}{8}$	$9\frac{7}{8}$	

The values of x are taken so as to differ from each other by the constant $\frac{1}{2}$; and the $\frac{1}{2}$ added to any value of x to obtain the next, may be called the *increment of* x. The increment may be taken either greater or less than $\frac{1}{2}$, as we please; also, the successive increments need not all be equal.

The general symbol Δx *is used to represent the increment of* x*; or, what is the same thing, the difference between two successive values of* x. It may be read "*difference of* x," or "*increment of* x"; but is usually read, briefly, "*delta* x."

Similarly, Δy *represents the increment of the function* y*; or, the difference between two successive values of the function* y. It is read "*delta* y."

It will be seen, in the table above, that equal increments given to x do not, necessarily, produce equal increments of y; nor do Δx and Δy have always the same algebraic sign.

If Δx is positive, x is, obviously, increased by adding Δx to it, whether x is positive or negative; hence, x is increasing when Δx is positive; and x is decreasing when Δx is negative. In like manner, y is increasing or decreasing according as Δy is positive or negative; and conversely. The locus of the function $y = x^3 - 6x^2 + 9x$ is shown in fig. 2, in the curve $OABC$. If $OM = x$ and $MP = y$

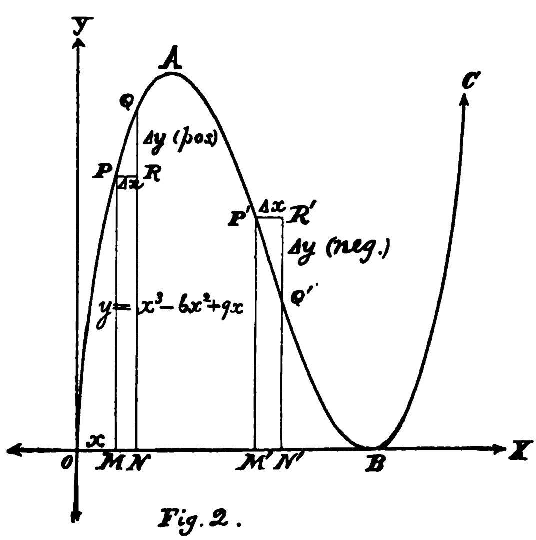

Fig. 2.

are the coördinates of P, and $MN = PR = \Delta x$, then $RQ = \Delta y$ is positive. Here y is increasing as x increases, and the curve is rising from P to Q. But if $OM' = x'$ and $M'P' = y'$ are the coördinates of P', then $M'N' = \Delta x$; and $R'Q' = \Delta y$ is negative. Here y is decreasing as x increases, and the curve is falling from P' to Q'.

In applying this rule about the increase, or decrease, of y, the student must be careful to see that no maximum value of y (as at point A) nor minimum value (as at point B), is included between $MP = y$ and $NQ = y + \Delta y$.

6. FORMULA FOR THE INCREMENT OF THE FUNCTION.

In the function of § 5, viz.,

$$(1) \qquad y = x^3 - 6x^2 + 9x,$$

let x_1 and y_1 be any pair of corresponding values of the variable and the function; that is,

$$(2) \qquad y_1 = x_1^3 - 6x_1^2 + 9x_1.$$

Add to x_1 the increment Δx, so that we have $x = x_1 + \Delta x$; then the function will assume a new value $y = y_1 + \Delta y$.

Equation (1) gives

$$(3) \qquad \begin{aligned} y_1 + \Delta y &= (x_1 + \Delta x)^3 - 6(x_1 + \Delta x)^2 + 9(x_1 + \Delta x) \\ &= x_1^3 - 6x_1^2 + 9x_1 + (3x_1^2 - 12x_1 + 9)\Delta x \\ &\qquad + (3x_1 - 6)(\Delta x)^2 + (\Delta x)^3. \end{aligned}$$

Subtracting (2) from (3) gives

$$(4) \qquad \Delta y = (3x_1^2 - 12x_1 + 9)\Delta x + (3x_1 - 6)(\Delta x)^2 + (\Delta x)^3.$$

This formula gives the increment, Δy, of the function, in terms of any assigned value, x_1, of the variable, and of the increment Δx, which has been given to that value of the variable. For example, if $x_1 = 0$, and $\Delta x = \frac{1}{2}$, then $\Delta y = 9(\frac{1}{2}) - 6(\frac{1}{2})^2 + (\frac{1}{2})^3 = 3\frac{1}{8}$; and if $x_1 = 1$, and $\Delta x = \frac{1}{2}$, then $\Delta y = -\frac{5}{8}$. (Compare the values of Δy given in the table of § 5.)

The general formula for the increment of any function

$$(5) \qquad y = f(x),$$

can be obtained in the same way; viz., let x_1 and y_1 be corresponding values of x and y in equation (5); then

(6) $y_1 = f(x_1)$.

Give to x the increment Δx, then y will take the increment Δy, and equation (5) gives

(7) $y_1 + \Delta y = f(x_1 + \Delta x)$;

and subtracting (6) from (7) gives

(8) $\Delta y = f(x_1 + \Delta x) - f(x_1)$.

Instead of a pair of fixed corresponding values x_1 and y_1 in $y = f(x)$, we might take any pair, as x and y; then the increment formula (8) would be

(9) $\Delta y = f(x + \Delta x) - f(x)$.

7. REMARK ON THE USE OF THE TERM "FUNCTION."

Whenever there is such a connection between two variables, that any variation in either causes a necessary corresponding variation in the other, and, when the assignment of a fixed value to one determines one or more corresponding values of the other, then they are said to be functions, the one of the other. The decision as to which one shall be called *the function*, and which, *the variable*, is entirely arbitrary; and is generally determined by convenience. For example, the area of a circle would be called a function of its radius, if, for any reason, it was desirable to treat the radius as independent variable; or, the radius would be called a function of the area, if it was desired to regard the area as independent variable. So, in any equation involving two variables, such as $a^2y^2 + b^2x^2 - a^2b^2 = 0$, where a and b are constants, either y may be called a function of x, if x is the independent variable; or x may be called a function of y, if it is desired to regard y as the independent variable.

In rectangular coördinates, the ordinate of any point of a fixed curve, is a function of the abscissa of the point; or, the abscissa is a function of the ordinate of the point.

However, the usual practice is to regard *the ordinate as function*, and the *abscissa as independent variable — called, briefly, variable.*

Exercises.

1. The area, y, of a circle (radius x) is $y = \pi x^2$. Plot the curve representing this function; find the formula for Δy; and calculate Δy when $x = 2$ and $\Delta x = .3$.

2. The space, s, traversed in time t, by a body falling in a vacuum, is $s = \frac{1}{2} g t^2$, g being constant and t expressed in seconds. Plot the curve which shows the relation between s and t; find Δs in terms of t and Δt; and find the increments of s, if $\Delta t = 1$, at the end of 3, 5, and 10, seconds.

3. The volume, v, of a sphere of radius r, is $v = \frac{4}{3} \pi r^3$. Plot the curve representing v as a function of r; find Δv in terms of r and Δr; and find the increment added to the volume, when $r = 3$ and the increment .5 is given to radius.

CHAPTER III.

DERIVED FUNCTIONS (DERIVATIVES): AND THEIR GEOMETRIC MEANING.

8. RATIO OF THE INCREMENT OF THE FUNCTION TO INCREMENT OF THE VARIABLE.

In the function

$$(1) \qquad y = x^3 - 6x^2 + 9x$$

of § 6, it has been shown that

$$(2) \qquad \Delta y = (3x^2 - 12x + 9)\,\Delta x + (3x - 6)\,(\Delta x)^2 + (\Delta x)^3,$$

where Δy is the increment of the function, produced by giving the variable x an increment Δx.

From equation (2), by division, the ratio of the increments is found to be

$$(3) \qquad \frac{\Delta y}{\Delta x} = 3x^2 - 12x + 9 + (3x - 6)\,\Delta x + (\Delta x)^2.$$

By assigning values to x and Δx, in equation (3), this ratio is easily calculated. It will, obviously, be positive if Δy and Δx have the same sign; and negative if their signs are unlike.

In general, for any function $y = f(x)$, as shown in equation (9) of § 6,

$$(4) \qquad \Delta y = f(x + \Delta x) - f(x);$$

whence,

$$(5) \qquad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

which is the general expression for the ratio of the increment of function to increment of variable.

9. THE DERIVED FUNCTION, OR DERIVATIVE.

In equations (2) and (3) of § 8, give x the fixed value x_1; and let the corresponding value of the function be y_1. Then we get

$$(1) \qquad \Delta y = (3x_1^2 - 12x_1 + 9)\,\Delta x + (3x_1 - 6)\,(\Delta x)^2 + (\Delta x)^3;$$

$$(2) \qquad \frac{\Delta y}{\Delta x} = 3x_1^2 - 12x_1 + 9 + (3x_1 - 6)\,\Delta x + (\Delta x)^2.$$

The increment Δx, being perfectly arbitrary, may be required to vary in any manner we choose. It should be noted, however, that the fixed quantities x_1 and y_1 are not changed, by causing Δx to change value. To fix the ideas, let us suppose Δx to start with some positive value, and then to assume, in successive numerical order, a set of values such that each shall be less than the one immediately preceding it, — that is, *let Δx be a positive, decreasing, variable quantity.* Let it be postulated, moreover, that this process of assigning values to Δx, is capable of such continuation that the values assigned to it shall finally become less than some fixed value, which can be chosen arbitrarily small, — that is, *let it be assumed that Δx can be made as small as we please.*

If Δx is made to vary in the manner just defined, equation (1) shows that the numerical value of Δy will decrease as we decrease Δx; and that *we can cause Δy to take as small a numerical value as we please, by choosing Δx small enough.*

This may be expressed by saying: *we can make Δy approach the *limit zero, by causing Δx to approach the limit zero:* or, it may be expressed more briefly, still, by saying: *Δy is an † infinitesimal if Δx is an infinitesimal.*

An inspection of equation (2) shows that the ratio $\dfrac{\Delta y}{\Delta x}$ will approach the limit $3x_1^2 - 12x_1 + 9$, as we make Δx approach the limit zero. This may be formally expressed by the equation

---

* When a variable quantity X, approaches in value a fixed quantity A, in such a way, that the numerical value of the difference $A - X$, can be made to become, and remain, less than a fixed positive quantity a_1, which may be chosen arbitrarily small, — then X is said to approach the *limit* A; or, A is called the *limit* of X. This may be formally expressed: lim. $[X] = A$.

† If the limit A, of any variable X, is zero, X is called an *infinitesimal*, — that is, *X is an infinitesimal if lim.* $[X] = 0$.

The student should note, carefully, the two essential properties of an infinitesimal, viz.:

a) it is a *variable*, b) it approaches the *limit zero.*

Zero is not an infinitesimal; for zero is a constant. A minutely small, fixed quantity, is *not* an infinitesimal; though it may be one of the values which the variable infinitesimal can take.

Values actually assumed by an infinitesimal, need not be, relatively, very small — it is only necessary that we shall be able to make them small, if we wish to; and, to make them as small as we please.

$$*(3) \qquad \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] = 3x_1^2 - 12x_1 + 9,$$

when $y = x^3 - 6x^2 + 9x$.

If we had used the value x_2 for x, instead of x_1, the result

$$(4) \qquad \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] = 3x_2^2 - 12x_2 + 9,$$

would have been obtained; that is, the limit of the ratio $\frac{\Delta y}{\Delta x}$ will have different values if different initial values of the variable x are taken; and this limit is determined in value when x is fixed. Hence, *the limit of the ratio* $\frac{\Delta y}{\Delta x}$ *is, in this example, a function of* x.

If we leave x unrestricted, we shall get [see equation (3) § 8]

$$(5) \qquad \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] = 3x^2 - 12x + 9,$$

which is the general value of the ratio $\frac{\Delta y}{\Delta x}$, when the function given is $y = x^3 - 6x^2 + 9x$.

This new function, $3x^2 - 12x + 9$, obtained from $y = x^3 - 6x^2 + 9x$ as the limit of the ratio $\frac{\Delta y}{\Delta x}$ when Δx approaches the limit zero, is called *the first derived function of the given primitive function*, $y = x^3 - 6x^2 + 9x$, *with respect to its variable* x. It is called, also, *the derivative, with respect to* x, *of the function* $y = x^3 - 6x^2 + 9x$.

A more compact symbol than $\lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right]$, which we shall use, is $D_x y$; that is,

$$\dagger(6) \qquad D_x y \equiv \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right].$$

Hence, if $y = x^3 - 6x^2 + 9x$,

$$(7) \qquad D_x y \equiv D_x(x^3 - 6x^2 + 9x) = 3x^2 - 12x + 9.$$

The general function, $y = f(x)$, may be treated in precisely the same manner as we have just now considered the special function $y = x^3 - 6x^2 + 9x$. From § 8, equation (5), we thus get, as *the*

---

* The symbol $\doteq$ means "*approaches the limit.*"

† The symbol $\equiv$ means "is identical with."

most general expression for the first derived function of the primitive function $y = f(x)$, the following:

$$(8) \qquad D_x y \equiv D_x f(x) = \lim_{\Delta x \doteq 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x.} \right].$$

The derived function of $y = f(x)$, defined by equation (8), is, in general, a new function of x, whose form will be completely fixed when we assign a form to $y = f(x)$; that is, *if a primitive function* $y = f(x)$ *is given, its derived function,* $D_x f(x)$, *is a determinate function of* x. This will be shown later for all the common, or ordinary, forms of $f(x)$. Every derived function may be calculated, directly, by using equation (8); but we shall soon develop rules for calculating derived functions, which, although based on equation (8), are much more direct and simple than the immediate application of (8) would be.

The process of obtaining the derived function from the primitive function, is called *differentiation* — *to differentiate* a given primitive function, is, *to find its derived function.*

The phrase "derived function" is generally shortened to *derivative.*

In the foregoing discussion, leading up to equation (8), Δx was restricted to positive values. This restriction is not, in general, necessary. The value of the limit of $\frac{\Delta y}{\Delta x}$, for a given value of x, say x_1, will, generally, be the same, whether Δx is taken positive or negative.

Exercises.

1. Calculate, by using equation (8), the following derivatives:

$a)$ $D_x x = 1$, $\qquad b)$ $D_x a = 0$,

$c)$ $D_x x^2 = 2x$, $\qquad d)$ $D_x a x^2 = 2ax$,

$e)$ $D_x(x^2 + b) = 2x$, $\qquad f)$ $D_x x^3 = 3x^2$.

2. If m is any positive integer, show that $D_x x^m = m x^{m-1}$.

3. Show that $D_x \sqrt{x} = \frac{1}{2\sqrt{x}}$.

4. If u is a function of x, and $f(u)$ is a function of u, show that

$$D_x f(u) = D_u f(u) D_x u.$$

5. If u is a function of x, show that $D_x a u = a D_x u$, a being a constant.

6. If u is a function of x, show that $D_x \sqrt{u} = \frac{1}{2\sqrt{u}} D_x u$.

7. Show that $D_x u^m = m u^{m-1} D_x u$, when u is a function of x, and m is $a)$ a positive integer; $b)$ a positive fraction; $c)$ a negative integer; $d)$ a negative fraction.

8. If u and v are functions of x, prove the following:

a) $D_x(u \pm v) = D_x u \pm D_x v,$

b) $D_x uv = u\, D_x v + v\, D_x u,$

c) $D_x\left(\frac{u}{v}\right) = \frac{v\, D_x u - u\, D_x v}{v^2}.$

9. Given that $\lim\limits_{\theta \doteq 0}\left[\frac{\sin\theta}{\theta}\right] = 1$, show that

a) $D_x \sin u = \cos u\, D_x u,$

b) $D_x \cos u = -\sin u\, D_x u,$

when u is a function of x.

10. By means of the results in exercises 8 (c) and 9, derive the following:

a) $D_x \tan u = \sec^2 u\, D_x u,$

b) $D_x \text{ctn}\, u = -\csc^2 u\, D_x u,$

c) $D_x \sec u = \sec u \tan u\, D_x u,$

d) $D_x \csc u = -\csc u\, \text{ctn}\, u\, D_x u,$

e) $D_x \text{vers}\, u = \sin u\, D_x u,$

f) $D_x \text{covers}\, u = -\cos u\, D_x u.$

11. Given that $\lim\limits_{z=\infty}\left[\left(1 + \frac{1}{z}\right)^z\right] = 2.7182818 + = e$, prove that

a) $D_x \log_a u = \frac{\log_a e}{u} D_x u$; and if $a = e$ this becomes

b) $D_x \log_e u = \frac{1}{u} D_x u.$

10. GEOMETRIC MEANING OF THE RATIO OF THE INCREMENT.

Let the curve AB, in fig. 3, be the locus of the equation $y = f(x)$; and let $OM_1 = x_1$, and $M_1P_1 = y_1 = f(x_1)$, be the coörds. of a fixed pt. P_1 on AB. Let $M_1N = \Delta x = P_1R$: then $ON = OM_1 + M_1N = x_1 + \Delta x$, and $NQ = f(x_1 + \Delta x)$.

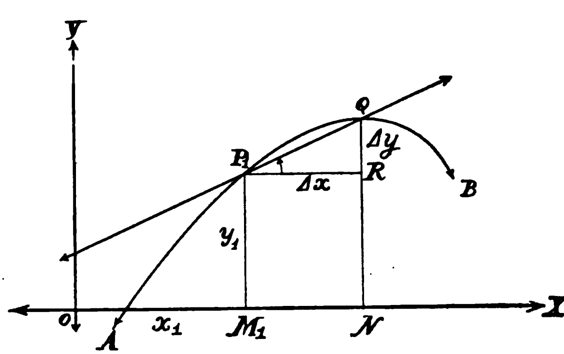

Fig. 3.

Hence, $\Delta y = RQ = NQ - NR = NQ - M_1P_1$
$$= f(x_1 + \Delta x) - f(x_1)$$
$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x} = \frac{RQ}{P_1R}.$$

Draw the secant P_1Q. Then P_1RQ is a right triangle, and

$$\tan \angle RP_1Q = \frac{RQ}{P_1R} = \frac{\Delta y}{\Delta x} = \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}.$$

Hence, the ratio of the increment of the function to the increment of the variable, expresses the slope of the secant joining the points (x_1, y_1) *and* $(x_1 + \Delta x, y_1 + \Delta y)$, *on the curve which represents the given function.*

Exercise.

Find the slope of the secant line joining the points of the curve

$$y = x^3 - 6x^2 + 9x \quad \text{(See fig. 2 of § 6)},$$

corresponding to $x_1 = \frac{1}{2}$ and $x_1 + \Delta x = \frac{3}{4}$. *Ans.* $2\frac{11}{16}$.

11. GEOMETRIC MEANING OF THE DERIVATIVE OF A GIVEN FUNCTION.

Let the curve AP_1QB be the locus of the given function $y = f(x)$. Then, as in § 10, it may be shown that

$$(1) \qquad \tan \angle RP_1Q = \frac{RQ}{P_1R} = \frac{\Delta y}{\Delta x} = \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}.$$

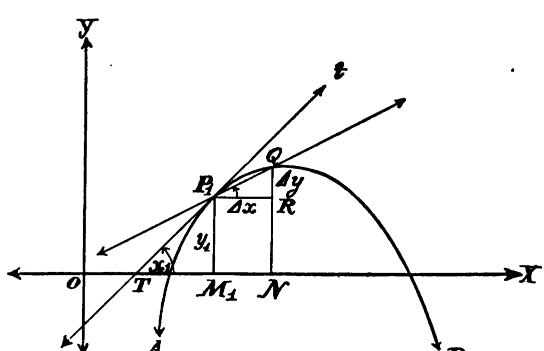

Fig. 4.

Under the conditions stated in the definition of the derivative, $M_1N \equiv \Delta x \doteq 0$ [see § 9, (8)]. This will cause the point Q to traverse the curve towards P_1, making the secant P_1Q rotate about P_1. Obviously, the secant will approach, as its limiting position, the line TP_1t, *which is the tangent to the curve at* P_1. Hence we get the following:

(2) $$\lim_{\Delta x \doteq 0}\left[\angle RP_1Q\right] = \angle RP_1t = \angle XTP_1; \text{ and}$$

(3) $$\lim_{\Delta x \doteq 0}\left[\tan \angle RP_1Q\right] = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]$$
$$= \lim_{\Delta x \doteq 0}\left[\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\right]$$
$$= D_x f(x)\Big|_{x = x_1} = \tan XTP_1.$$

Therefore, the value of the derivative of $f(x)$, when x_1 is substituted for x, is the slope of the tangent to the curve $y = f(x)$, at the point whose abscissa is x_1.

So we may show that, in general, $D_x f(x)$ *is, for a given value of* x, *geometrically represented by the slope of the curve* $y = f(x)$, *at the point on the curve whose abscissa is the given value of* x.

The phrase "slope of a curve at a given point" means, the slope of its tangent at that point.

For illustration, take the function $f(x) = x^3 - 6x^2 + 9x$. Here we have found that $D_x f(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. (The locus of $y = x^3 - 6x^2 + 9x$ is shown in fig. 2.)

If $x = 1$, or $x = 3$, $D_x f(x) = 0$; and the curve is parallel to the x-axis.

If $x < 1$, $D_x f(x)$ is positive; and the curve, in this region, makes an acute angle with the x-axis.

If $1 < x < 3$, $D_x f(x)$ is negative; and in this region (from A to B in fig. 2), the curve makes an obtuse angle with the x-axis.

If $x > 3$, $D_x f(x)$ is positive; and in this region (from B on to the right), the curve makes an acute angle with the x-axis.

Exercises.

1. Determine where the curve $y = \sin x$ *a*) is parallel to, *b*) makes an acute angle with, *c*) an obtuse angle with, the x-axis.

Can this curve be perpendicular to the x-axis?

2. Where does the curve $y = \sqrt{x}$ make an acute angle with the x-axis? Find the slope at $x_1 = 1$, — at $x_2 = 4$?

3. Where does the curve $y = \sqrt{4 - x^2}$ make an obtuse angle with the x-axis? What is its slope at $x_1 = -1$; — at $x_2 = 1$; — at $x_3 = 2$?

4. What is the slope of the curve $y = \frac{1}{x}$ at $x_1 = -2$; — at $x_2 = -1$; — at $x_3 = 0$; — at $x_4 = 1$; — at $x_5 = 3$?

Plot the curve.

CHAPTER IV.

GEOMETRIC APPLICATIONS OF DERIVATIVES: CONTINUOUS VARIATION OF VARIABLE AND FUNCTION: TESTS FOR INCREASING OR DECREASING FUNCTIONS.

12. TANGENTS, NORMALS, SUBTANGENTS AND SUBNORMALS.

Let the curve AB be the locus of the function $y = f(x)$, or of $f(x, y) = 0$.

Let P_1 be some fixed point on AB, having the coördinates $OM_1 = x_1$ and $M_1P_1 = y_1$. Let TP_1t be the tangent, and NP_1n be the normal

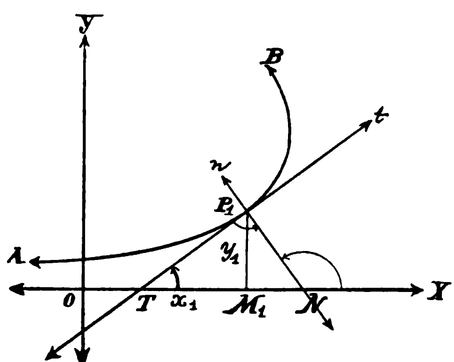

Fig. 5.

at P_1. If the tangent and normal at P_1 meet the x-axis in T and N, respectively, then the projection TM_1, of TP_1, upon the x-axis, is called the *subtangent* of P_1; and the projection M_1N, of P_1N, on the x-axis, is called the *subnormal* of P_1.

From § 11 we get $\tan XTP_1 = D_x f(x) \Big|_{x = x_1}$:

hence, from Analytical Geometry, the equation of the tangent TP_1t is

$$(1) \qquad y - y_1 = D_x f(x) \Big|_{x = x_1} (x - x_1).$$

Also, $\angle XNP_1 = \angle NP_1T + \angle XTP_1 = 90° + \angle XTP_1$:

$$\therefore \tan XNP_1 = \tan(90° + XTP_1) = -\operatorname{ctn} XTP_1$$

$$= \frac{-1}{\tan XTP_1} = \frac{-1}{D_xf(x)\Big|_{x=x_1}}.$$

Hence, from Anal. Geom., the equation of the normal NP_1n is

$$(2) \qquad y - y_1 = \frac{-1}{D_xf(x)\Big|_{x=x_1}}(x - x_1).$$

By Trigonometry, we easily get, from the right triangles TM_1P_1 and M_1NP_1, $TM_1 = \frac{M_1P_1}{\tan M_1TP_1}$, and $M_1N = \frac{M_1P_1}{\tan M_1NP_1}$.

But $\qquad M_1P_1 = y_1, \quad \tan M_1TP_1 = D_xf(x)\Big|_{x=x_1},$

and $\quad \tan M_1NP_1 = \operatorname{ctn} XTP_1 = \frac{1}{\tan XTP_1} = \frac{1}{D_xf(x)\Big|_{x=x_1}}.$

Whence we get the following formulæ for the subtangent and subnormal at P_1:

$$(3) \qquad \textit{Subtangent} = TM_1 = \frac{y_1}{D_xf(x)\Big|_{x=x_1}},$$

$$(4) \qquad \textit{Subnormal} = M_1N = y_1 D_xf(x)\Big|_{x=x_1}.$$

The foregoing formulæ, for equations of tangent and normal at P_1, and for the lengths of the subtangent and subnormal, are perfectly general; and may be used for all functions for which the value of $D_xf(x)\Big|_{x=x_1}$ can be calculated.

Exercises.

1. Find the equations of the tangent and normal of the following curves, at the point specified on each.

a) $y = 2x^2 - 5x$	where	$x_1 = 3$,
b) $y = 4x^3 - 7$	"	$x_1 = 2$,
c) $y = \frac{1}{x}$	"	$x_1 = -\frac{1}{2}$,
d) $y^2 = 2mx + nx^2$	"	$x_1 = m$.

2. Find the lengths of the subtangent and subnormal, for each curve in ex. 1, at the point given on each.

13. ANGLE OF INTERSECTION OF TWO CURVES.

If two curves intersect, their angle of intersection is the angle formed by their tangents at their common point.

Let two curves, AP_1B and CP_1D, intersect at P_1; and let TP_1 and RP_1 be their tangents, at the common point P_1. Then their angle of intersection is $\angle TP_1R$.

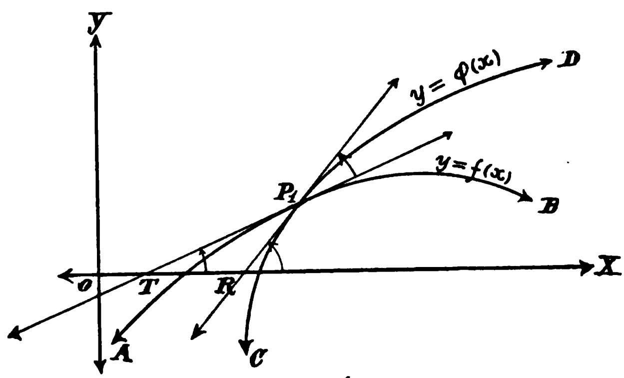

Fig. 6.

But $\angle TP_1R = \angle XRP_1 - \angle XTP_1$: hence

$$(1) \qquad \tan TP_1R = \tan(XRP_1 - XTP_1) = \frac{\tan XRP_1 - \tan XTP_1}{1 + \tan XRP_1 \tan XTP_1}.$$

Suppose the equations of the curves AP_1B and CP_1D are $y = f(x)$ and $y = \phi(x)$, respectively: then, from § 11, we get

$$\tan XTP_1 = D_x f(x)\Big|_{x=x_1}; \text{ and } \tan XRP_1 = D_x \phi(x)\Big|_{x=x_1}.$$

Substituting in (1) gives

$$(2) \qquad \tan TP_1R = \frac{D_x\phi(x)\Big|_{x=x_1} - D_x f(x)\Big|_{x=x_1}}{1 + D_x\phi(x)\Big|_{x=x_1} D_x f(x)\Big|_{x=x_1}},$$

which is a general formula for the angle between two intersecting curves.

Exercises.

1. Find the angle, — or angles, when they intersect in more than one point, — of intersection of the following curves:

$$a)\ y = 2x \qquad \text{and} \qquad y^2 = 8x,$$

$$b)\ y^2 = 4ax \qquad \text{“} \qquad y = \frac{4a}{x^2},$$

$$c)\ y = \frac{2}{x} \qquad \text{“} \qquad y^2 = x^2 - 9.$$

2. Show that the length of that part of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the coördinate axes, is constant, and equal to a.

3. Find the angle of intersection of the curves:

$$a)\ y = \sin x \qquad \text{and} \qquad y = \cos x,$$

$$b)\ y = \sin x \qquad \text{“} \qquad y = \tan x,$$

$$c)\ y = \cos x \qquad \text{“} \qquad y = \tan x,$$

14. CONTINUOUS VARIATION OF VARIABLE AND FUNCTION.

The variable x is said to vary *continuously*, from a value x_0 to a value x_n, when it takes, in the order of numerical sequence, the series of values

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ \ldots\ x_n;$$

and when the successive differences, or increments,

$$x_1 - x_0,\ x_2 - x_1,\ x_3 - x_2,\ \ldots\ x_n - x_{n-1},$$

are all infinitesimal. (See definition of infinitesimal.)

The successive increments $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2$, etc., are not necessarily equal; but, as they may be arbitrarily chosen, it is possible to make them all equal, — that is, we may, for simplicity, take $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = \Delta x$.

We may say, therefore, that *the variable x changes value continuously, from a fixed value x_0 to a fixed value x_n, if it assumes, successively, the set of values*

$$x_0,\ x_0 + \Delta x,\ x_0 + 2\Delta x,\ x_0 + 3\Delta x,\ \ldots\ x_0 + (n-1)\Delta x,\ x_n,$$

which have the common increment Δx; and when Δx can be made arbitrarily small.

If $x_0 < x_n$, then x is increasing; and all the increments are positive: but if $x_0 > x_n$, then x is decreasing, and its increments are negative.

The *continuous* variation of x, from x_0 to x_n, as above defined, is said to cause a corresponding *continuous* variation of the function $y = f(x)$, from the fixed value $y_0 = f(x_0)$ to $y_n = f(x_n)$, when each of the increments

$$f(x_1) - f(x_0),\ f(x_2) - f(x_1),\ f(x_3) - f(x_2),\ \ldots f(x_n) - f(x_{n-1}),$$

is an infinitesimal.

If the increments of x are equal, we may put the increments of the function in the forms $\quad f(x_0 + \Delta x) - f(x_0)$,

$$f(x_0 + 2\Delta x) - f(x_0 + \Delta x),\quad f(x_0 + 3\Delta x) - f(x_0 + 2\Delta x),\quad \text{etc.}$$

The definition of *continuous* variation of the function $y = f(x)$, as x varies from x_0 to x_n, may be expressed: *$y = f(x)$ varies continuously, from $y_0 = f(x_0)$ to $y_n = f(x_n)$, if $\Delta y = f(x + \Delta x) - f(x)$ is an infinitesimal (Δx being an infinitesimal) for every value of x from x_0 to x_n, including x_0 and x_n.*

The function $\tan x$ fails to satisfy the foregoing definition at $x = \frac{\pi}{2}$; the function $\frac{1}{1-x}$ fails to satisfy the definition at $x = 1$; — these functions are said to be *discontinuous* at the values of x specified.

The function $\tan^{-1}\left(\frac{1}{x}\right) - \frac{x}{1+x^2}$ is discontinuous at $x = 0$; for, if $x < 0$, and $x \doteq 0$, the function approaches the value $-\frac{\pi}{2}$: but, if $x > 0$, and $x \doteq 0$, the function approaches $+\frac{\pi}{2}$. In this case the discontinuity is *finite*.

The ordinary functions are rarely discontinuous, except for single, isolated, values of the variable. For example, $\tan x$ is discontinuous only at the values $x = n\frac{\pi}{2}$, where n is any odd integer.

15. TEST FOR AN INCREASING OR DECREASING FUNCTION.

Suppose the variable x to increase from a fixed value x_0 to the fixed value x_n $(x_0 < x_n)$. The function may, (1) increase all the way, (2) decrease all the way, or (3) part of the way increase, and part of the way decrease. For example, $y = \sqrt{a^2 - x^2}$ increases from 0 to a, as x increases from $-a$ to 0; but decreases from a to 0, as x increases from 0 to a. The function $y = \sin x$ some-

times increases, and sometimes decreases, if x varies from $x_0 = 0$ to $x_n = 2\pi$. (See § 4.)

We wish to establish a test by which we can determine whether, at a given value x_1, the function $y = f(x)$ is increasing or decreasing, as x increases from x_1 to $x_1 + \Delta x$, Δx being a positive infinitesimal increment. (See fig. 3.)

The increment of y, at this value of x, is $\Delta y = f(x_1 + \Delta x) - f(x_1)$.

If $f(x_1 + \Delta x) > f(x_1)$, then $y = f(x)$ is increasing, as x passes from x_1; and $\Delta y = f(x_1 + \Delta x) - f(x_1)$ is positive.

Conversely, if $f(x_1 + \Delta x) - f(x_1) = \Delta y$ is positive, then

$$f(x_1 + \Delta x) > f(x_1),$$

and $y = f(x)$ is increasing, as x increases from x_1 to $x_1 + \Delta x$. These conditions hold true, however small Δx and Δy become. Hence, we get the condition:

$$(1) \qquad D_x y \Big|_{x = x_1} = \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x}\right] \Bigg|_{x = x_1}$$

is positive if $y = f(x)$ is increasing, as x increases from the value x_1: and, conversely, if $D_x y \big|_{x = x_1}$ is positive, $y = f(x)$ is increasing, as x increases from x_1.

So it may be shown that, throughout any continuous range of values of x, all of which make $D_x f(x)$ positive, the function $y = f(x)$ will increase as x increases; and, therefore, decrease as x decreases.

For example, take the function $y = \sin x$. Here, $D_x y = \cos x$, which is positive if $-\frac{\pi}{2} < x < \frac{\pi}{2}$; hence, *$\sin x$ increases while x increases* from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$; and, *$\sin x$ decreases if x decreases* from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. (See § 4.)

In like manner it may be shown that, if $D_x f(x)$ remains negative, throughout a continuous range of values of x, then $f(x)$ *decreases as x increases;* and *increases as x decreases.*

For example, $D_x \sin x = \cos x$ is negative if $\frac{\pi}{2} < x < \frac{3\pi}{2}$; hence, $\sin x$ decreases as x increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. (See § 4.)

A few simple curves, constructed by the student, will show, graphically, that when any curve is rising from left to right (that is, the function increases as the variable increases), its slope is positive: and, if it is falling from left to right, the slope will be negative.

Exercises.

1. Determine *a*) the ranges of values of x which make $y = 2x^3 + 3x^2 - 36x$ increase as x increases; and *b*) the range of values of x which make it decrease as x increases.

2. Determine the values of x, at which the above function changes from decreasing to increasing; or, *vice versa*.

3. Show that the function $y = 2\sqrt{px}$ increases, always, as x increases.

4. Determine the limits of x between which the locus of the function $y = \frac{b}{a}\sqrt{x^2 - a^2}$ is *descending* from left to right; — is *ascending* from left to right.

5. Show that the function $y = \tan x$ increases, always, as x increases.

6. How do the following functions vary, as x increases:

a) $y = e^x$, *b*) $y = \sin^{-1} x$,
c) $y = \tan^{-1} x$, *d*) $y = \log x$?

CHAPTER V.

SUCCESSIVE DERIVED FUNCTIONS, OR SUCCESSIVE DERIVATIVES.

16. THE SECOND DERIVED FUNCTION, AND HIGHER DERIVED FUNCTIONS.

Consider the function $y = x^m$, when m is some positive integer.

Its first derived function is $D_x x^m = m x^{m-1}$.

Obviously, $m x^{m-1}$ is a function of x, and can be differentiated. We get $\quad D_x D_x x^m \equiv D_x m x^{m-1} = m(m-1) x^{m-2}$.

This expression, $m(m-1)x^{m-2}$, is *the first derived function of the first derived function;* and is called the *second derived function* of the primitive function, x^m. It is denoted by the symbol

$$D_x^2 x^m \equiv D_x D_x x^m.$$

In like manner, we get the *third derived function* of x^m to be

$$D_x^3 x^m \equiv D_x D_x^2 x^m = D_x m(m-1) x^{m-2} = m(m-1)(m-2) x^{m-3}.$$

The *fourth derived function* of x^m is

$$D_x^4 x^m \equiv D_x D_x^3 x^m = m(m-1)(m-2)(m-3) x^{m-4}.$$

Similarly, the m^{th} *derived function* of x^m is

$$D_x^m x^m = \underline{|m}; \text{ and, the } (m+1)^{th} \text{ is } D_x^{m+1} x^m = 0.$$

All higher derived functions of x^m, m being a positive integer, are zero. In this example the differentiation terminates at the m^{th} derived function. It is not thus with all functions.

Exercises.

1. Find the fourth derived functions of the following

$a)\ y = (1+x)^m, \qquad c)\ y = (1+x^2)^m,$

$b)\ y = (1-x)^m, \qquad d)\ y = (1-x^2)^m.$

2. Show that

$$D_x \sin x = \sin\left(x + \frac{\pi}{2}\right), \qquad D_x^2 \sin x = \sin\left(x + \frac{2\pi}{2}\right),$$

. . . .

$$D_x^m \sin x = \sin\left(x + \frac{m\pi}{2}\right).$$

3. Show that $D_x^m \cos x = \cos\left(x + \frac{m\pi}{2}\right)$.

[In examples 2 and 3 the differentiation can be carried on indefinitely.]

4. Plot the following curves:

a) $y = \sin x$, c) $y = D_x^2 \sin x = -\sin x$,

b) $y = D_x \sin x = \cos x$, d) $y = D_x^3 \sin x = -\cos x$.

[Place these four curves on the same axes, using a large enough unit to make a good, open, figure.]

17. AN INTERPRETATION OF THE ALGEBRAIC SIGN OF THE SECOND DERIVED FUNCTION.

Let $y = f(x)$ be the primitive function. It has been shown that if $D_x f(x)$ is positive, throughout a given range of the values of x, then $f(x)$ increases as x increases over that range; and that when $D_x f(x)$ is negative, throughout a given range of x, then $f(x)$ decreases as x increases.

Now, the first derived function, $D_x f(x)$, is *the primitive function of the second derived function*, $D_x^2 f(x)$; hence it follows that $D_x f(x)$ is increasing or decreasing, throughout a given range of values of x, as x increases over that range, according as $D_x^2 f(x)$ remains positive or negative.

For example, if $y = \sin x$ is the primitive function, $D_x \sin x = \cos x$, and $D_x^2 \sin x = -\sin x$. Then, $D_x^2 \sin x$ is negative if x has any value from 0 to π; and is positive if x has any value from π to 2π; which shows that $D_x \sin x = \cos x$ is increasing from $x = 0$ to $x = \pi$, and is decreasing from $x = \pi$ to $x = 2\pi$, — as we know, from Trigonometry, to be the fact.

In like manner, *the sign of the third derived function*, for any given range of values of x, will determine whether the second derived function is increasing, or decreasing, throughout that range.

Similarly, *the sign of the m^{th} derived function*, will determine whether the $(m-1)^{th}$ derived function is increasing, or decreasing, as x increases.

Exercises.

1. Plot the curves representing the primitive $y = x^2 - 5x + 4$, its first, and second derived functions. Place them on the same axes; and show that the first derived function increases from $x = -\infty$ to $x = +\infty$.

2. Plot the curves representing the primitive $y = 2x^3 - 15x^2 + 24x$, and its first and second derived functions, — placing the three curves on the same axes.

Show that the primitive function increases from $x = -\infty$ to $x = 1$; decreases from $x = 1$ to $x = 4$; and increases from $x = 4$ to $x = +\infty$. Show, also, that $D_x y$ decreases from $x = -\infty$ to $x = \frac{5}{2}$; and increases from $x = \frac{5}{2}$ to $x = +\infty$.

[Note that the curve representing the first derived function *crosses the x-axis for values of x at which the primitive function changes from increasing to decreasing*; or, *vice versa*. Note, in this respect, the curves traced in ex. 4 of § 16.]

CHAPTER VI.

MAXIMUM AND MINIMUM VALUES OF FUNCTIONS: APPLICATIONS.

18. MAXIMUM VALUES OF A CONTINUOUS, SINGLE-VALUED, FUNCTION.

Let $y = f(x)$ be the given function. Let x_1 be a value of x which makes the first derivative vanish; that is, $D_x f(x)\Big|_{x=x_1} = 0$. Then, at the point P_1, (x_1, y_1), of the locus of $y = f(x)$, the tangent must be parallel to the x-axis; since (§ 11) $D_x f(x)\Big|_{x=x_1}$ is the slope of the tangent at P_1. One of the cases in which the tangent is parallel to the x-axis, is shown in the adjoining figure, in which the curve

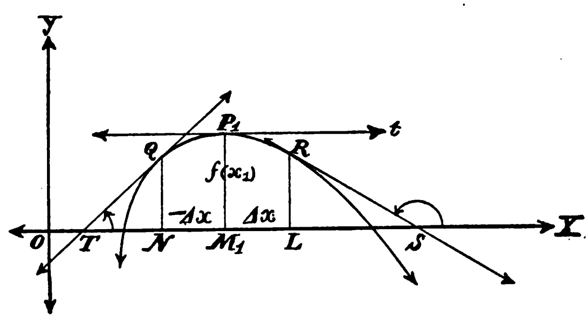

Fig. 7.

QP_1R represents the function $y = f(x)$. In this case, the ordinates of the curve immediately on each side of the point P_1, are, obviously, shorter than the ordinate of P_1, where the tangent P_1t is parallel to the x-axis: that is, the value of the function at P_1, $f(x_1) = y_1 = M_1P_1$, is greater than either of the values $f(x_1 - \Delta x) = NQ$, or $f(x_1 + \Delta x) = LR$. Hence, the value $f(x_1) = M_1P_1$ is called a **maximum** value of $f(x)$.

The further tests for a maximum are obtained by noting that $NQ = f(x)$ is increasing as Q approaches P_1; that is, as x increases

up to x_1: and, that $LR = f(x)$ is decreasing as R recedes from P_1; that is, as x increases above x_1. From § 15, $D_x f(x)$ is positive when $f(x)$ is increasing (x being supposed to increase), and negative when $f(x)$ is decreasing. Hence, as x increases from ON, through $OM_1 = x_1$ up to OL, $D_x f(x)$ decreases from the positive value $\tan XTQ$, through zero, to the negative value $\tan XSR$. Then, since $D_x f(x)$ is decreasing at $x = x_1$, its derivative, $D_x^2 f(x)$, must be negative at $x = x_1$.

Conversely, it may be shown that if

$$(1) \qquad D_x f(x) \Big|_{x = x_1} = 0, \quad \text{and}$$

$$(2) \qquad D_x^2 f(x) \Big|_{x = x_1} \quad \text{is negative,}$$

then $y_1 = f(x_1)$ *is a maximum value* of $y = f(x)$; for, since $D_x^2 y$ is negative at $x = x_1$, $D_x y$ is decreasing. But $D_x y \Big|_{x = x_1} = 0$; and, therefore, assuming $D_x y$ to be continuous, $D_x y \Big|_{x < x_1}$ must be positive, and $D_x y \Big|_{x > x_1}$ must be negative. Hence, $y = f(x) \Big|_{x < x_1}$ must be increasing as x increases, and $y = f(x) \Big|_{x > x_1}$ must be decreasing as x increases. If, therefore, conditions (1) and (2) are satisfied at a fixed value x_1 of x, for any continuous function $y = f(x)$, then the function will have a maximum value when this value x_1 is substituted for x. For example, if $y = \sin x$, then $D_x y = \cos x = 0$ and $D_x^2 y = -\sin x = -1$ when $x = \frac{\pi}{2}$: hence, $y = \sin x$ has a maximum value when $x = \frac{\pi}{2}$.

In the foregoing, it is assumed that both $y = f(x)$, and $D_x y$, are finite and continuous for $x = x_1$; as well as for values of x just below, and just above, x_1. The student will have little difficulty in detecting the cases, when the function and derivative are not continuous.

It is interesting to notice that, at a maximum value of $y = f(x)$, the first derivative curve, $y = D_x f(x)$, *crosses the x-axis, falling obliquely from left to right;* since $D_x f(x) \Big|_{x = x_1} = 0$, and $D_x^2 f(x) \Big|_{x = x_1}$ is negative.

It may be seen, from an easy investigation, that the preceding conditions for a maximum of $y = f(x)$, apply equally to the case shown in fig. 8, when the curve is below the x-axis; that is, when $y_1 = f(x_1)$ is negative. In this case, since NQ, M_1P_1, and LR, are negative, it is still true that $NQ < M_1P_1 > LR$.

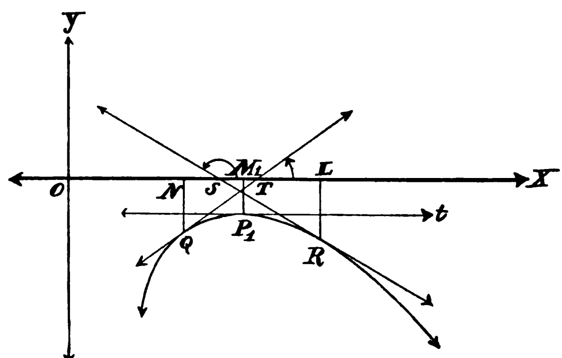

Fig. 8.

Problems.

1. Find the edges and volume of the open box of greatest content, which can be made from a sheet of metal a inches square, by cutting equal squares from the corners and bending up the sides.

2. Solve problem 1 when a rectangular sheet of metal, a inches long and b inches wide, is used instead of a square.

3. Find the base, altitude, and area, of the rectangle of maximum area, inscribed in the triangle whose base is b and altitude is a.

4. The $\begin{bmatrix}\text{strength}\\\text{stiffness}\end{bmatrix}$ of a beam of rectangular cross section, being proportional to the product of its breadth by the $\begin{bmatrix}\text{square}\\\text{cube}\end{bmatrix}$ of its depth, find the breadth and depth of the $\begin{bmatrix}\text{strongest}\\\text{stiffest}\end{bmatrix}$ beam that can be cut from a cylindrical log, $2a$ inches in diameter.

5. If the common slant height of a series of right cones is a, find the radius, altitude, volume, and semi-vertical angle, of that one having the greatest volume.

6. Show that the square has the greatest perimeter and area of all rectangles nscribed in a given circle.

7. In a voltaic cell the electromotive force $= f$ and internal resistance is r. Suppose the external resistance is R; then the power given out is

$$P = \frac{f^2 R}{(r + R)^2}.$$

Assuming f and r constants, and R as independent variable, find what relation exists between r and R, when the power P is a maximum.

8. Find the radius and altitude of the right circular cylinder of maximum volume, that can be inscribed in a sphere whose radius is a.

9. An open gutter, whose cross-section is an isosceles trapezoid, is to be made of a sheet of copper $a + 2b$ inches wide: the width of the base being a inches, and the breadth of the equal sloping sides being b inches. Find the width, across the top, which will make the carrying capacity of the gutter a maximum.

Show that, if $b = a$, the width of the top is twice the width of the bottom.

10. A strip of metal, $2a$ inches wide, is to be used in making an open gutter, whose cross-section is a segment of a circle. Find the angle subtended at the center by the arc $2a$, and the width across the top, when the carrying capacity of the gutter is a maximum.

19. MINIMUM VALUES OF A CONTINUOUS, SINGLE-VALUED, FUNCTION.

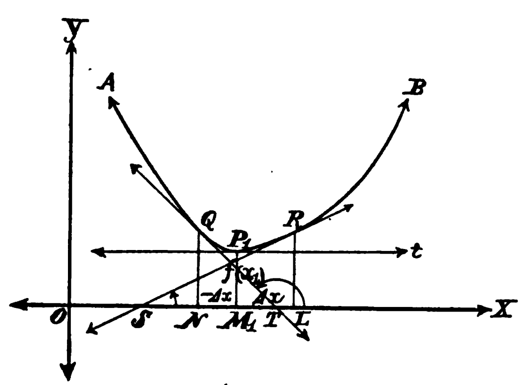

Fig. 9.

Let $y = f(x)$ be the given function, and let QP_1R be its locus in rectangular coördinates. It can be readily shown, as in § 18, that if

$$(1) \qquad D_x f(x) \Big|_{x = x_1} = 0, \text{ and}$$

$$(2) \qquad D_x^2 f(x) \Big|_{x = x_1} \text{ is positive,}$$

then the ordinate $M_1P_1 = y_1 = f(x_1)$ is less than either the ordinate $NQ = f(x_1 - \Delta x)$, or $LR = f(x_1 + \Delta x)$, immediately before and after x reaches x_1.

For, if $D_x^2 f(x)\Big|_{x=x_1}$ is positive, then $D_x f(x)$ is increasing as x increases through x_1. But, $D_x f(x)\Big|_{x=x_1} = 0$: hence, assuming $D_x f(x)$ to be continuous, $D_x f(x)\Big|_{x<x_1} = \tan XTQ$ is *negative;* and $D_x f(x)\Big|_{x>x_1} = \tan XSR$ is *positive.* Therefore, $f(x)\Big|_{x<x_1}$ is *decreasing*, and $f(x)\Big|_{x>x_1}$ is *increasing.* That is, if conditions (1) and (2), above, are satisfied, then

$$f(x_1 - \Delta x) > f(x_1) < f(x_1 + \Delta x).$$

In this case, $f(x)$ is said to have a **minimum value** when $x = x_1$.

As in the case of a maximum, the function may be negative; that is, its curve may lie below the x-axis; yet the same tests, (1) and (2), will apply, as may easily be proved.

Note that, at a minimum of $y = f(x)$, the first derivative curve $y = D_x f(x)$ *crosses the x-axis obliquely upward toward the right;* since $D_x f(x)\Big|_{x=x_1} = 0$, and $D_x^2 f(x)\Big|_{x=x_1}$ is positive.

Other tests for maximum and minimum values of $y = f(x)$, which apply to cases that occur rarely, will be given in section 20.

Problems.

1. A rectangular box, open at the top, is to be constructed so as to contain a given volume V. Find its base and altitude when the least possible material, of given thickness, is used in its construction.

2. Solve problem 1 when the top is to be closed.

3. A body, of weight P, is drawn along a horizontal plane by a force y, whose line of action makes an angle x with the plane. If the friction of unit weight, perpendicular to the plane, is f, show that y will be a minimum when $\tan x = f$; and, then $y = \dfrac{Pf}{\sqrt{1+f^2}}$.

4. A ball rolls down a smooth inclined plane, which makes an angle x with the horizontal plane. The time, t, required to pass over a given *horizontal* distance, a, is given by the formula

$$\tfrac{1}{2} g t^2 \sin x = a \sec x.$$

What angle x will make t a minimum?

5. A wall a feet high is b feet from a house. Find the length of the shortest ladder that will reach the house from the ground outside the wall; and the distance from the foot of the ladder to the bottom of the wall.

6. A series of n equally careful observations, giving the n results a_1, a_2, a_3, a_4 . . . a_n, not all equal, were made to determine an unknown magnitude x. The errors of the several observations are, obviously,

$$x - a_1,\ x - a_2,\ x - a_3,\ \ldots\ x - a_n;$$

and it is equally probable that any given result, a_k, is in excess or default; so that any one of the above differences may be either positive or negative. In any case, that value of x is, probably, most nearly correct which makes the sum of the squares of the errors, viz.:

$$y = (x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2 + \ \ldots\ + (x - a_n)^2,$$

a minimum: find this value of x.

7. Within an angle BAC, is given a point P, through which it is required to draw a straight line, so that the area of the triangle enclosed shall be a minimum. Show that the line is bisected at P.

8. What are the most economical dimensions of a cylindrical tin cup, holding a given quantity Q, the cup being open at the top.

9. What are the most economical dimensions of a closed cylindrical tin can, required to hold a given volume V.

10. Find the coördinates of the point on the curve $y = \frac{a}{x}\sqrt{ax - x^2}$ at which the subtangent is a minimum.

CHAPTER VII.

CUSPS: POINTS OF INFLEXION.

20. CUSP MAXIMA AND MINIMA.

The adjoining curves, AP_1B and CP_2D, show that it is possible for a single-valued continuous function $y=f(x)$, to have a maximum value (at P_1); or a minimum value (at P_2), when the derivative $D_xf(x)$ is infinite; that is, when the tangent to the curve (P_1t_1 or P_2t_2) is perpendicular to the x-axis.

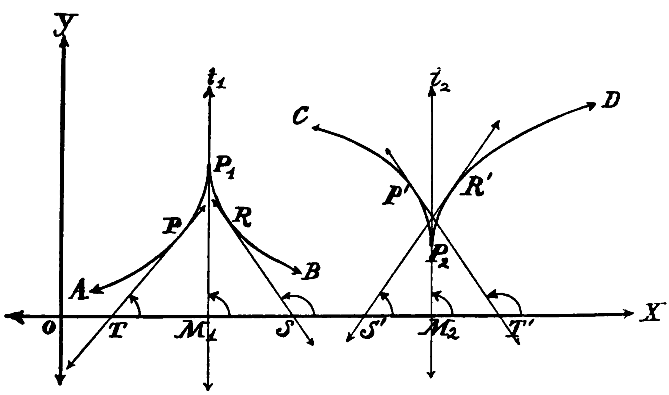

Fig. 10.

At the *cusp maximum* P_1, the following conditions are readily seen to hold:

$$(1) \qquad D_xf(x)\Big|_{x=x_1} = \infty,$$

$$(2) \qquad D_xf(x)\Big|_{x<x_1} = \tan XTP \text{ is positive,}$$

$$(3) \qquad D_xf(x)\Big|_{x>x_1} = \tan XSR \text{ is negative.}$$

At the *cusp minimum* P_2, we see that the following conditions are satisfied:

$$(4) \qquad D_x f(x) \Big|_{x=x_1} = \infty,$$

$$(5) \qquad D_x f(x) \Big|_{x<x_1} = \tan XT'P' \text{ is negative,}$$

$$(6) \qquad D_x f(x) \Big|_{x>x_1} = \tan XS'R' \text{ is positive.}$$

Whenever, therefore, a finite, single-valued, and continuous function $y=f(x)$ satisfies, for any given value x_1, the conditions (1), (2) and (3), $f(x_1)$ is a *maximum;* and if it satisfies conditions (4), (5) and (6), $f(x_1)$ is a *minimum.*

These cases are not common.

21. POINTS OF INFLEXION.

At the point P_1 on the curve APP_1RB, which is the locus of a single valued function $y=f(x)$, the *curvature is reversed.* Such a point as P_1 is called a *point of inflexion.*

At P_1 the curve crosses its tangent P_1t; and P_1t is called the *inflexion tangent.*

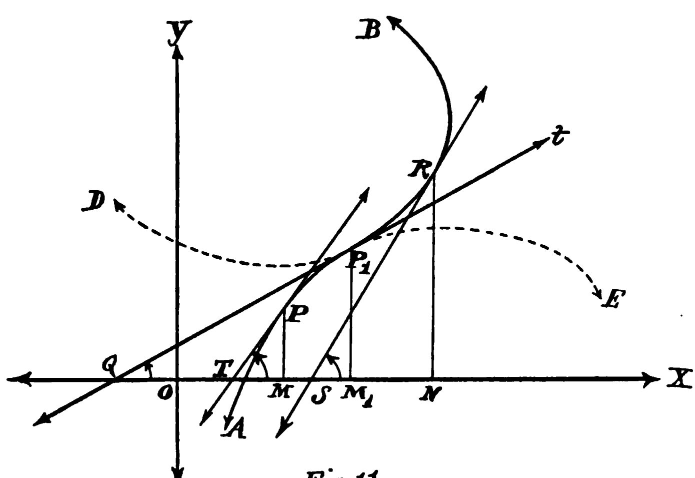

Fig. 11.

Let TP be a tangent to the curve at P, such that the abscissa, x, of P is less than the abscissa x_1 of P_1. Let x increase up to x_1;

the angle XTP, evidently, *decreases to the ultimate value* XQP_1; then, as x increases beyond x_1 the angle XSR, *increases from* XQP_1. In short, *the slope of the curve is a minimum at* P_1; that is, $D_x f(x)$ *is a minimum at* P_1.

If the locus of $y = f(x)$ had the position of the dotted curve DP_1E, it could, in like manner, be shown that $D_x f(x)$ has a *maximum at* P_1.

The characteristic property of a point of inflexion is, therefore, this: —

At a point of inflexion, on the locus of a single-valued, continuous function $y = f(x)$, *the slope*, $D_x f(x)$, *is a maximum or a minimum.*

It may be shown, conversely, that a point at which the slope is a maximum or minimum, is a point of inflexion.

Hence, the points of inflexion must be found by applying to the first derivative, $D_x f(x)$, the tests of §§ 18, 19 and 20 for maximum and minimum values.

In general, therefore, the conditions for a point of inflexion are

$$(1) \qquad D_x^2 f(x) \Big|_{x = x_1} = 0,$$

$$(2) \qquad D_x^3 f(x) \Big|_{x = x_1} \text{ is not zero.}$$

It may be noted that, if the inflexion-tangent is parallel to the x-axis, we have a case in which $D_x f(x) \Big|_{x = x_1} = 0$, but $y = f(x)$ has neither a maximum nor a minimum value. In this case,

$$D_x^2 f(x) \Big|_{x = x_1} = 0, \quad D_x^3 f(x) \Big|_{x = x_1} \text{ is not zero;}$$

and it may be seen that the first derivative curve will *touch the x-axis* for $x = x_1$, instead of crossing it as noticed in the cases of maxima and minima.

Exercises.

1. Show that the locus of any single-valued function $y = f(x)$ has its *concave side downwards, wherever* $D_x^2 f(x)$ *is negative:* and that its *concave side is upwards wherever* $D_x^2 f(x)$ *is positive.*

[SUGGESTION: — *Show that* $D_x f(x)$ *decreases, or increases, according as the concave side of the curve is down, or up.*]

2. Show that if $D_x f(x) \Big|_{x=x_1} = 0$, $D_x^2 f(x) \Big|_{x=x_1} = 0$, and $D_x^3 f(x) \Big|_{x=x_1} = 0$, then $f(x) \Big|_{x=x_1}$ will be a *maximum if* $D_x^4 f(x) \Big|_{x=x_1}$ *is negative:* and a *minimum if* $D_x^4 f(x) \Big|_{x=x_1}$ *is positive.*

3. Show that, at $x = x_1$ in $y = f(x)$, if the last derivative which vanishes is of *odd order* (as the 3rd or 5th), then $f(x)$ will be a maximum, or a minimum, according as the next higher derivative is negative, or positive.

4. Show that, at $x = x_1$ in $y = f(x)$, if the last derivative to vanish is of *even order* (as the 2nd or 4th), then $f(x)$ will have neither a maximum nor a minimum: but its locus will have a *point of inflexion* at $x = x_1$.

5. Show that the curve $y = \sin x$ has a point of inflexion at $x_1 = 0$; also, that $y = \tan x$ has a point of inflexion at $x_1 = 0$.

6. Find the point of inflexion; also the maximum and minimum values; of the function $y = x^3 - 6x^2 + 9x$. [See fig. 2.]

CHAPTER VIII.

DIFFERENTIALS OF FUNCTION AND VARIABLE.

22. DIFFERENTIALS.

Let the curve AB in fig. 12 be the locus of the continuous, and single-valued, function $y = f(x)$. Let $P_1(x_1, y_1)$ be a point on AB; and let P_1t be the tangent at P_1.

Give x the value $ON = OM_1 + M_1N = x_1 + M_1N$; and y will take the corresponding value $NQ = NR + RQ = y_1 + RQ$.

Let S be the point of intersection of the tangent P_1t with the ordinate NQ: the coördinates of S are $ON = OM_1 + P_1R = x_1 + P_1R$, and $NS = NR + RS = y_1 + RS$.

We have given (see § 5) to M_1N and RQ the names, respectively, *increment of* x and *increment of* y; and have used the symbols Δx and Δy to represent them. They are *the differences between the coördinates of the two points* P_1 *and* Q, *on the curve* AB.

Now, P_1R ($= M_1N$) and RS are *the differences between the coördinates of the two points* P_1 *and* S, *on the tangent* P_1t. We shall use the symbols dx, and dy, to represent them; that is $P_1R = dx$, and $RS = dy$; and shall call them, respectively, the **differential of** x, and the **differential of** y.

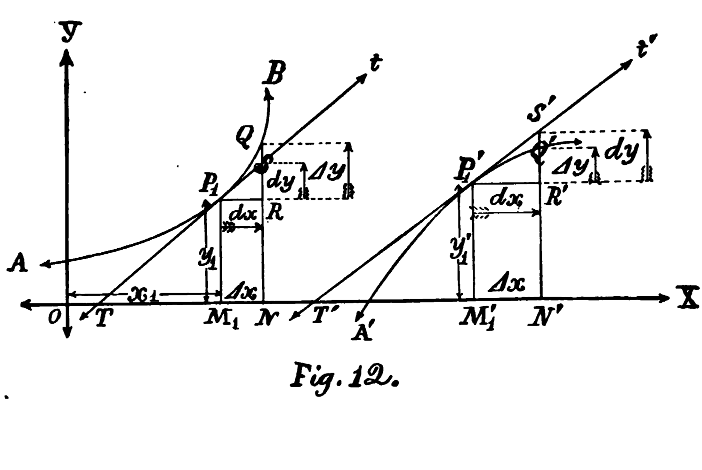

Fig. 12.

It will be seen that, since $P_1R = M_1N$,

$$(1) \qquad dx = \Delta x;$$

that is, *the differential of x, and the increment of x, are equal.*

It will be seen, also, that RS and RQ cannot, in general, be equal. If, as in fig. 12, the curve P_1Q lies *above* its tangent P_1S, then $RS < RQ$, or $dy < \Delta y$: but if, as may happen, the curve between P_1 and Q should lie *below* its tangent at P_1, then $dy > \Delta y$. The student can readily convince himself by making suitable figures, that, except for special values of dx, *the only case in which dy is equal to Δy, is when the locus of $y = f(x)$ is a straight line;* that is, when $y = f(x)$ is of the form $y = mx + b$.

Hence, in general, we find that

$$(2) \qquad dy \gtrless \Delta y.$$

Since dx and Δx are equal, *dx can be regarded as the increment of x;* and may be substituted for Δx, if we choose, in every formula in which Δx appears.

But dy and Δy are not the same; though an inspection of fig. 12 will show that both dy and Δy are dependent infinitesimals; both approaching zero, simultaneously, when dx is made to approach zero. This may be seen better by noting that, in the right triangle, P_1RS, $RS = \tan RP_1S \times P_1R$; that is,

$$(3) \qquad dy = \tan RP_1S \cdot dx:$$

so that $dy \doteq 0$ as $dx \doteq 0$; since, if P_1 is fixed, $\tan RP_1S$ is constant, when dx is made to vary.

Equation (3) shows, what may be seen also from fig. 12, that dy depends not only on dx, but upon $\tan RP_1S$ as well: so that, for equal values of dx, dy will take different values at different points on the curve, since $\tan RP_1S$ will vary upon all loci but the straight line.

We have shown (see § 11) that, for any point of the curve $y = f(x)$, the slope of the tangent is expressed by the value which the derivative D_xy assumes at that point.

Then we may substitute for $\tan RP_1S$ in (3) its value $D_xy \Big|_{x = x_1}$, and get

$$(4) \qquad dy = D_xy \Big|_{x = x_1} dx.$$

Equation (4) gives the exact relation between dy and dx *at the fixed point* P_1. If we do not restrict the point, but take P *any point on the locus of* $y = f(x)$; then, the relation between dy and dx, at P, will be exactly expressed by the equation

$$(5) \qquad dy = D_x y\, dx, \quad \text{or} \quad df(x) = D_x f(x)\, dx;$$

in which it is understood that the particular value of $D_x y$, at P, is to be substituted.

Hence dy can be calculated, for a given increment dx, whenever $D_x y$ can be calculated.

From equation (5) we get

$$(6) \qquad \frac{dy}{dx} = D_x y:$$

so that we may use the ratio $\frac{dy}{dx}$ as a new symbol to represent the derived function of $y = f(x)$.

We shall now extend the meaning of the terms "differentiation" and "to differentiate", *to include the operation of finding the differential of a given function;* as well as the operation of finding the derivative. Hereafter, *to differentiate a function*, will generally mean, *to find the differential of the function.*

As shown in equation (5) *the differential of* $f(x)$ *is obtained by multiplying the derivative of* $f(x)$ *by the differential of* x.

CHAPTER IX.

FORMULÆ FOR DIFFERENTIATING ORDINARY FUNCTIONS.

23. FUNDAMENTAL DIFFERENTIAL FORMULÆ.

From the results of exercises at the end of § 9, we may derive the following differential formulæ by using equation (5) of § 21 : —

I. $d(u \pm v) = du \pm dv$,

II. $duv = u\,dv + v\,du$,

III. $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$,

IV. $dau = a\,du$,

V. $da = 0$,

VI. $df(u) = D_u f(u)\,du$,

VII. $du^m = mu^{m-1}\,du$,

VIII. $d\sqrt{u} = \frac{1}{2\sqrt{u}}\,du$,

IX. $d\log_a u = \log_a e \cdot \frac{du}{u}$,

X. $d\log_e u = \frac{du}{u}$,

XI. $d\sin u = \cos u\,du$,

XII. $d\cos u = -\sin u\,du$,

XIII. $d\tan u = \sec^2 u\,du$,

XIV. $d\operatorname{ctn} u = -\csc^2 u\,du$,

XV. $d\sec u = \sec u \tan u\,du$,

XVI. $d\csc u = -\csc u \operatorname{ctn} u\,du$,

XVII. $d\operatorname{vers} u = \sin u\,du$,

XVIII. $d\operatorname{covers} u = -\cos u\,du$.

In the above formulæ, u and v may be any "ordinary" functions of x. In particular, we may put $u = x$ in any of them. For example, putting $u = x$ in X and XI gives $d\log_e x = \frac{dx}{x}$, and $d\sin x = \cos x\,dx$, respectively. So each of the others may be treated.

The formulæ above are sufficient for the differentiation of algebraic, trigonometric, and logarithmic functions. We need, still, formulæ for differentiating the exponential, and anti-trigonometric, functions. These we shall now deduce.

Let $v = a^u$; where a is any positive constant, and u is any function of x. Taking logarithms of both members, we get $\log_e v = u \log_e a$. Taking differentials of this we get

$$(1) \qquad d\log_e v = \log_e a\,du.$$

But, by X, $d\log_e v = \frac{1}{v}dv = \frac{1}{a^u}da^u$.

Substituting in (1), and clearing of fractions, gives

$$(2) \qquad da^u = a^u \log_e a\, du.$$

Again, let $v = \sin^{-1}u$; then $\sin v = u$ and $d\sin v = du$. But $d\sin v = \cos v\, dv = \cos[\sin^{-1}u]\, d\sin^{-1}u = \sqrt{1-u^2}\, d\sin^{-1}u$. $\therefore \sqrt{1-u^2}\, d\sin^{-1}u = du$, whence,

$$(3) \qquad d\sin^{-1}u = \frac{du}{\sqrt{1-u^2}}.$$

In like manner the differentials of $\cos^{-1}u$, $\tan^{-1}u$, etc., may be deduced. These formulæ are collected below. Formulæ XXII to XXVIII may be worked as exercises by the student.

XIX. $da^u = a^u \log_e a\, du$.

XX. $de^u = e^u du$ (if $a = e$).

XXI. $d\sin^{-1}u = \frac{du}{\sqrt{1-u^2}}$.

XXII. $d\cos^{-1}u = \frac{-du}{\sqrt{1-u^2}}$.

XXIII. $d\tan^{-1}u = \frac{du}{1+u^2}$.

XXIV. $d\operatorname{ctn}^{-1}u = \frac{-du}{1+u^2}$.

XXV. $d\sec^{-1}u = \frac{du}{u\sqrt{u^2-1}}$.

XXVI. $d\csc^{-1}u = \frac{-du}{u\sqrt{u^2-1}}$.

XXVII. $d\operatorname{vers}^{-1}u = \frac{du}{\sqrt{2u-u^2}}$.

XXVIII. $d\operatorname{covers}^{-1}u = \frac{-du}{\sqrt{2u-u^2}}$.

The foregoing twenty-eight formulæ will be found sufficient for obtaining the differentials of the most complicated of the ordinary functions. Each one may be transformed into the corresponding derivative formula, by dividing by dx. For example, from XXI we get $\frac{d\sin^{-1}u}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$; or, $D_x \sin^{-1}u = \frac{1}{\sqrt{1-u^2}} D_x u$.

If we put $u = x$ in this, it becomes

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}; \quad \text{or} \quad D_x \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$$

So each of the others may be treated.

It is worth while to note that the addition of any constant to a given function makes no change either in its derivative, or in its differential. For example,

$$D_x[\sin x + K] = D_x \sin x + D_x K = D_x \sin x;$$

since $D_x K = 0$ when $K =$ any constant. Also,

$$d[\sin x + K] = d\sin x + dK = d\sin x;$$

since $dK = 0$ when $K =$ any constant. So, in general,

$$D_x[f(x) + K] = D_x f(x) + D_x K = D_x f(x);$$

and $d[f(x) + K] = df(x) + dK = df(x)$; since $D_x K$ and dK both equal zero.

Exercises.

1. $d\left(\frac{1}{m}\sin mx\right) = \cos mx\,dx.$

2. $d\left(\frac{-1}{m}\cos mx\right) = \sin mx\,dx.$

3. $d\log\sin x = \cot x\,dx.$

4. $d\log\sec x = \tan x\,dx.$

5. $d\left(\frac{1}{a}\tan ax\right) = \sec^2 ax\,dx.$

6. $d\left(\frac{-1}{a}\operatorname{ctn} ax\right) = \csc^2 ax\,dx.$

7. $d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{a^2 - x^2}}.$

8. $d\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{dx}{a^2 + x^2}.$

9. $d\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{dx}{x\sqrt{x^2 - a^2}}.$

10. $d\log(x + \sqrt{x^2 + a^2}) = \frac{dx}{\sqrt{x^2 + a^2}}.$

11. $d\log(x + \sqrt{x^2 - a^2}) = \frac{dx}{\sqrt{x^2 - a^2}}.$

12. $d\left[\frac{1}{2a}\log\frac{a+x}{a-x}\right] = \frac{dx}{a^2 - x^2}.$

13. $d\operatorname{vers}^{-1}\frac{x}{a} = \frac{dx}{\sqrt{2ax - x^2}}.$

14. $d\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right] = \sqrt{a^2 - x^2}\,dx.$

15. $d\left[\frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log(x + \sqrt{x^2 \pm a^2})\right] = \sqrt{x^2 \pm a^2}\,dx.$

16. $d\left[\frac{1}{a}\cos^{-1}\frac{a}{x}\right] = \frac{dx}{x\sqrt{x^2 - a^2}}.$

17. $d\left[\frac{1}{a}\log\left(\frac{x}{a + \sqrt{a^2 \pm x^2}}\right)\right] = \frac{dx}{x\sqrt{a^2 \pm x^2}}.$

18. $d\sqrt{x^2 - a^2} = \frac{x\,dx}{\sqrt{x^2 - a^2}}.$

19. $d\left[\frac{1}{3}\sqrt{(x^2 \pm a^2)^3}\right] = x\sqrt{x^2 \pm a^2}\,dx.$

20. $d\sqrt{\frac{x-1}{x+1}} = \frac{dx}{(x+1)\sqrt{x^2 - 1}}.$

21. $d\,[\sin^{-1}x - \sqrt{1-x^2}] = \sqrt{\dfrac{1+x}{1-x}}\,dx.$

22. $d\,[x\sin^{-1}x + \sqrt{1-x^2}] = \sin^{-1}x\,dx.$

23. $d\left[\dfrac{x^2}{2}(\log x - \tfrac{1}{2})\right] = x\log x\,dx.$

24. $d\left[\dfrac{1}{a}e^{ax}\left(x - \dfrac{1}{a}\right)\right] = xe^{ax}\,dx.$

25. $d\log\tan\dfrac{x}{2} = \csc x\,dx.$

26. $d\left[\tfrac{1}{2}\log\dfrac{1-\cos x}{1+\cos x}\right] = \csc x\,dx.$

27. $d\tan\dfrac{x}{2} = \dfrac{dx}{1+\cos x}.$

28. $d\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right) = \dfrac{dx}{1-\sin x}.$

29. $d\,\mathrm{ctn}\,\dfrac{x}{2} = \dfrac{-\,dx}{1-\cos x}.$

30. $d\left[\dfrac{x}{2} - \tfrac{1}{2}\sin x\cos x\right] = \sin^2 x\,dx.$

31. $d\left[\dfrac{x}{2} + \tfrac{1}{2}\sin x\cos x\right] = \cos^2 x\,dx.$

32. $d\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right) = \sec x\,dx.$

CHAPTER X.

ANTI-DIFFERENTIALS: ANTI-DIFFERENTIATION: THEOREMS.

24. ANTI-DIFFERENTIALS.

In Trigonometry we have become acquainted with the following symbols:

(1) $y = \sin x$, and $x = \sin^{-1} y$;

(2) $y = \tan x$, and $x = \tan^{-1} y$; etc., etc.

The forms $\sin x$, $\tan x$, etc., are called *Trigonometric Functions;* and the forms $\sin^{-1} y$, $\tan^{-1} y$, etc., are called *Anti-Trigonometric Functions.*

It should be noticed that the two equations $y = \sin x$, and $x = \sin^{-1} y$, mean precisely the same thing; and that the difference is, solely, in the symbolic way of expressing it; the first form expressing that *y is the sine of the angle x;* the second, that *x is the angle whose sine is y.* Each form is useful, and necessary: the first, to designate the sine when the angle is the datum; and the second, to denote the angle when the sine is given.

So, in the Calculus, we have become acquainted with the following symbolic expressions:

(3) $$d\left[\frac{x^{n+1}}{n+1} + K\right] = x^n dx,$$

(4) $$d[\log x + K] = \frac{dx}{x},$$

(5) $$d[e^x + K] = e^x dx,$$

(6) $$d[\sin x + K] = \cos x dx,$$ etc.

These are called *differentials.* The function enclosed in the bracket is the *datum*, and the right hand member of the equation is the *result.* The symbol of differentiation, d, may be regarded either as asking the question: What is the differential of the datum? or, as an *operator*, expressing the operation of calculating the differential of the datum.

In many problems, particularly in applications of the Calculus, *the differential of some unknown function will be the datum; and the function itself will be the result desired.* In such case, when the differential of a function is the datum but the function is unknown, — when the function has to be found from its differential, — we shall call the unknown function the **anti-differential** of the given differential. For example, we get from equations (3) to (6), above, the following:

(7) Anti-differential of $x^n dx$ is $\dfrac{x^{n+1}}{n+1} + K$,

(8) " " $\dfrac{dx}{x}$ is $\log x + K$,

(9) " " $e^x dx$ is $e^x + K$,

(10) " " $\cos x dx$ is $\sin x + K$.

The operation of calculating the anti-differential from a given differential, we shall call **anti-differentiation.** This operation is the inverse, or the opposite, of the direct operation of differentiation.

The symbol used to indicate anti-differentiation is $\int$. *Call it "long S."* For example, in place of the words "anti-differential of," as in equations (7) to (10) above, we shall write $\int$, as follows:

$$(11) \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + K,$$

$$(12) \qquad \int \frac{dx}{x} = \log x + K,$$

$$(13) \qquad \int e^x dx = e^x + K,$$

$$(14) \qquad \int \cos x dx = \sin x + K.$$

The symbol $\int$ may be regarded either *as asking a question*, or *as an operator*. The *question* may be stated in either of two equivalent ways: (a) What is the anti-differential of the datum? or (b) What is the function which has the datum for its differential? The right hand members of equation (11) to (14) are, respectively, the

answers to this question, when the given differential is that written after $\int$. As an *operator*, $\int$ expresses the operation of calculating the anti-differential of the given differential.

To sum up, we may say: *The anti-differential of a given differential, is that function whose differential is given.*

Obviously, every differential formula, or result, will furnish a correlative anti-differential formula.

It should be observed that *differentiation* furnishes a single, or unique, result; but that *anti-differentiation* furnishes a result that may contain an arbitrary constant K: so that the result obtained by the latter operation is not unique. For example, we get, by differentiation, the single result $d[\sin x + K] = \cos x\,dx$; but we get by anti-differentiation the result $\int \cos x\,dx = \sin x + K$, where K may be any constant whatsoever.

So, every anti-differential may consist of two parts, viz.: a function of x, plus a constant. For if

$$(15) \qquad d\phi(x) = f(x)\,dx, \text{ then}$$

$$*(16) \qquad \int f(x)\,dx = \phi(x) + K;$$

because
$$\begin{aligned} d[\phi(x) + K] &= d\phi(x) + dK \\ &= d\phi(x) \\ &= f(x)\,dx, \text{ by (15).} \end{aligned}$$

We shall find that it will not be difficult to determine K, in problems involving anti-differentiation.

The geometric meaning of K may be seen by considering the simple case $dy = m\,dx$. This gives by anti-differentiation, $y = mx + K$; which, for different values of K, will be represented by different lines, all having the same slope $m = \frac{dy}{dx}$; and all, therefore, parallel.

---

* It is assumed, here, that, *the differential being given, its anti-differential exists.* As a general statement this requires proof: but the student will have no hesitation in accepting it for those cases in which he actually discovers the anti-differential, whose correctness he can verify by differentiation. He may reserve his doubts for those cases in which he cannot discover the anti-differential, after he has tried all the means and methods which are yet to be presented to him.

In like manner, the loci of $y = \phi(x)$ and $y = \phi(x) + K$, which have equal differentials, may be called **parallel curves**; for, at a given value x_1 on any two curves of the series, say $y = \phi(x) + K_1$ and $y = \phi(x)$, we get $y_1 = \phi(x_1) + K_1$ and $y_2 = \phi(x_1)$; hence, $y_1 - y_2 = K_1$. Also, $\left.\frac{dy}{dx}\right|_{x=x_1}$ has the same value on both curves. From these two results it follows that: (*a*) *all distances, measured parallel to the y-axis, between points having the same abscissa, are equal; and* (*b*) *the slopes of the curves at points having the same abscissa, are the same.*

25. THE OPERATORS d AND $\int$, WHEN APPLIED SUCCESSIVELY, CANCEL EACH OTHER.

For, by definition, if

$$(1) \qquad d[\phi(x) + K] = f(x)\,dx, \text{ then,}$$

$$(2) \qquad \int f(x)\,dx = \phi(x) + K.$$

Now, operating with $\int$ on both members of (1) gives

$$(3) \qquad \int d[\phi(x) + K] = \int f(x)\,dx.$$

Comparing (2) and (3) shows that

$$(4) \qquad \int d[\phi(x) + K] = \phi(x) + K.$$

Equation (4) shows that the operation d, upon $\phi(x) + K$, followed by the operation $\int$, leaves $\phi(x) + K$ unchanged. Again, operating with d, on both members of (2), gives

$$(5) \qquad d\int f(x)\,dx = d[\phi(x) + K]$$
$$= f(x)\,dx, \text{ from (1).}$$

Hence, the operation $\int$ upon $f(x)\,dx$, followed by the operation d, leaves $f(x)\,dx$ unchanged.

This holds good, of course, only for those cases in which the anti-differential is known to exist.

26. TWO GENERAL THEOREMS ON ANTI-DIFFERENTIALS.

A. *Any constant factor may be placed on either side of the symbol* $\int$ *without affecting the result; that is,*

(1) $$\int (av)\,dx = a\int v\,dx.$$

Let $du = v\,dx$; then, § 25,

(2) $$u = \int v\,dx.$$

Now, by IV, § 23,

(3) $$a\,du = d(au);$$

whence, by substituting in (3) the values of u and du, we get

(4) $$av\,dx = d\left[a\int v\,dx\right].$$

Getting anti-differentials of both members of (4), and using § 25, we find

(5) $$\int (av)\,dx = a\int v\,dx.$$

Q. E. D.

B. *The anti-differential of the sum of a set of differentials, is equal to the sum of the anti-differentials of the several terms;* that is,

(6) $$\int [u\,dx \pm v\,dx] = \int u\,dx \pm \int v\,dx.$$

Let $dU = u\,dx$, and $dV = v\,dx$; then $U = \int u\,dx$, and $V = \int v\,dx$.

It has been shown (§ 23, I) that

(7) $$d[U \pm V] = dU \pm dV.$$

Hence, we get, by substituting the values of U and V,

(8) $$d\left[\int u\,dx \pm \int v\,dx\right] = u\,dx \pm v\,dx.$$

Getting anti-differentials of both members of (8) we find, by § 25,

(9) $$\int u\,dx \pm \int v\,dx = \int [u\,dx \pm v\,dx],$$

which is the same as (6), above; and proves theorem (B) when the sum consists of two terms.

In like manner the proof can be extended to any case in which the number of terms is finite; but if the set of terms is an infinite series, theorem (B) is not always applicable.

Exercises.

1. Find the following anti-differentials:

a) $\int dx$, *b*) $\int a\,dx$, *c*) $\int 2x\,dx$,

d) $\int x\,dx$, *e*) $\int ax\,dx$, *f*) $\int 3x^2\,dx$,

g) $\int (a+x)\,dx$, *h*) $\int (a-x)\,dx$.

2. Find the following anti-differentials:

a) $\int \frac{dx}{x}$, *b*) $\int e^x\,dx$, *c*) $\int \cos x\,dx$,

d) $\int \frac{dx}{2\sqrt{x}}$, *e*) $\int \sec^2 x\,dx$, *f*) $\int \operatorname{cosec}^2 x\,dx$,

g) $\int \sec x \tan x\,dx$, *h*) $\int \operatorname{cosec} x \cot x\,dx$,

i) $\int \frac{dx}{\sqrt{1-x^2}}$, *j*) $\int \frac{dx}{1+x^2}$.

CHAPTER XI.

METHODS OF ANTI-DIFFERENTIATION: FORMULÆ.

27. ANTI-DIFFERENTIATION: FUNDAMENTAL FORMS OF ANTI-DIFFERENTIALS.

Differentiation of the ordinary functions, as we have seen, is always possible; and, unless the function is extremely complicated, the operation is readily performed.

But anti-differentiation is an operation which can not in every case be performed. Indeed, if a differential is given at random, it is quite as likely that its anti-differential cannot be found, as that it can. This is due to the fact that there are other functions than the ordinary functions; and that the differentials of some of these other functions are ordinary functions.

To effect anti-differentiation, the student should be perfectly familiar with the fundamental differential forms, as given in Chap. IX, *so as to be able to recognize the anti-differential, if possible, by inspecting the given differential.* For many important problems, whose solution depends upon anti-differentiation, it is quite possible, either to recognize the anti-differential by inspection, or, to transform the given differential so that its anti-differential may be recognized. The most common of these transformations are: (1) transferring a constant factor from one side to the other of the $\int$; (2) multiplying and dividing the given differential by some well chosen quantity; (3) adding and subtracting the same quantity; and (4) changing the variable by substituting another having a known connection with the first. When these simpler methods fail, others may enable us to find the desired anti-differential, or function. The latter methods will be the subjects of future sections. In this section we shall present the primary anti-differential formulæ, as the basis of all work in anti-differentiation; and show how they may be employed in treating various differential forms, which can, after slight transformations, be compared directly with some one of the primary forms, thus to suggest the desired anti-differential result.

The following may be termed the fundamental anti-differentials; they may be verified by differentiating both members of each equation; or, they may be derived from the corresponding differentials given in Chap. IX:

I. $\int u^m du = \dfrac{u^{m+1}}{m+1}$, or $\int x^m dx = \dfrac{x^{m+1}}{m+1}$, $m \lessgtr -1$;

*II. $\int \dfrac{du}{u} = \log u$, or $\int \dfrac{dx}{x} = \log x$;

III. $\int \dfrac{du}{\sqrt{u}} = 2\sqrt{u}$, or $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$;

IV. $\int e^u du = e^u$, or $\int e^x dx = e^x$;

V. $\int a^u du = \dfrac{a^u}{\log a}$, or $\int a^x dx = \dfrac{a^x}{\log a}$;

VI. $\int \sin u\, du = -\cos u$, or $\int \sin x\, dx = -\cos x$;

VII. $\int \cos u\, du = \sin u$, or $\int \cos x\, dx = \sin x$;

VIII. $\int \sec^2 u\, du = \tan u$, or $\int \sec^2 x\, dx = \tan x$;

IX. $\int \operatorname{cosec}^2 u\, du = -\cot u$, or $\int \operatorname{cosec}^2 x\, dx = -\cot x$;

X. $\int \sec u \tan u\, du = \sec u$, or $\int \sec x \tan x\, dx = \sec x$;

XI. $\int \operatorname{cosec} u \cot u\, du = -\operatorname{cosec} u$, or $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$;

XII. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\dfrac{u}{a}$, or $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\dfrac{x}{a}$;

XIII. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a}$, or $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$;

XIV. $\int \dfrac{du}{u\sqrt{u^2 - 1}} = \sec^{-1} u$, or $\int \dfrac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$;

XV. $\int \dfrac{du}{\sqrt{2u - u^2}} = \operatorname{vers}^{-1} u$, or $\int \dfrac{dx}{\sqrt{2x - x^2}} = \operatorname{vers}^{-1} x$;

XVI. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\log\dfrac{u-a}{u+a}$, if $u > a$;

---

* In this, as in all other formulæ involving logarithms, the base $e = 2.71828+$ is understood unless another base is stated.

XVII. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log [u + \sqrt{u^2 \pm a^2}];$

XVIII. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1}\frac{u}{a};$

XIX. $\int \sqrt{u^2 \pm a^2}\, du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \log [u + \sqrt{u^2 \pm a^2}].$

The foregoing anti-differential formulæ are not all fundamental, in the sense that no one of them is derivable from some other; but they include the fundamental ones, as well as others which are of frequent occurrence.

The forms containing u are more general, and often more useful, than the second forms containing x, which are the special cases of the first in which the function $u = x$.

An arbitrary constant K is to be understood with each of the above forms. Thus, the complete form of $\int e^u du$ is $e^u + K$.

In attempting to find an anti-differential, the student should first compare the given differential with the fundamental forms above; if it is identical with some one of them, the result is evident; but if it is not, then he must investigate whether he can find such a transformation of the given differential as will make it identical either with some one of them, or with some other known form.

Exercises.

1. Verify forms XII, XIII, XVI, XVII, XVIII, and XIX, in the foregoing list, by differentiation.

2. Prove the following by the use of the proper forms from those given above.

a) $\int \frac{dx}{x^2} = \frac{-1}{x};$

b) $\int e^{3x} dx = \frac{1}{3} e^{3x};$

c) $\int x^{-\frac{3}{2}} dx = -2x^{-\frac{1}{2}};$

d) $\int \frac{dx}{x+a} = \log (x + a);$

e) $\int (a + bx)^2 dx = \frac{(a+bx)^3}{3b};$

f) $\int \frac{x^3 dx}{a + bx^4} = \frac{1}{4b} \log (a + bx^4);$

g) $\int \tan x\, dx = -\log \cos x = \log \sec x;$

h) $\int \cot x\, dx = \log \sin x;$

i) $\int \frac{dx}{\sin x \cos x} = \log \tan x;$

j) $\int \frac{dx}{\sin x} = \log \tan \frac{x}{2};$

k) $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2} + x\right)} = \log \tan\left[\frac{\pi}{4} + \frac{x}{2}\right]$;

l) $\int \sin mx\, dx = \frac{-1}{m} \cos mx$; *m*) $\int \cos mx\, dx = \frac{1}{m} \sin mx$;

n) $\int \sin (a + bx)\, dx = -\frac{1}{b} \cos (a + bx)$.

3. Prove $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$ if $x < a$.

SUGGESTION: $\frac{1}{a^2 - x^2} \equiv \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right]$.

4. Prove $\int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1}\left[\frac{2ax + b}{\sqrt{4ac - b^2}}\right]$, when $b^2 < 4ac$.

SUGGESTION: $ax^2 + bx + c \equiv a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$.

5. Prove $\int \frac{dx}{ax^2 + bx + c} = \frac{-2}{2ax + b}$, when $b^2 = 4ac$.

6. Prove $\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \log\left[\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right]$, when $b^2 > 4ac$.

SUGGESTION: $ax^2 + bx + c \equiv a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$.

7. Prove $\int \frac{(ax + b)\, dx}{x^2 + px + q} = \frac{a}{2} \log [x^2 + px + q] + \frac{2b - ap}{2} \int \frac{dx}{x^2 + px + q}$.

SUGGESTION: $ax + b \equiv \frac{a}{2}(2x + p) + \frac{1}{2}(2b - ap)$.

NOTE: the anti-differential, $\int \frac{dx}{x^2 + px + q}$, can be found by ex. 4, 5, or 6, according as $p^2 < 4q$, $p^2 = 4q$, or $p^2 > 4q$.

28. ANTI-DIFFERENTIATION "BY SUBSTITUTION" OF A NEW VARIABLE.

a) Differentials involving $\sqrt{a^2 - x^2}$ can sometimes be transformed, so as to be easily handled, by substituting $x = a \sin \phi$, or $x = a \cos \phi$. In the first case $dx = a \cos \phi\, d\phi$, and $\sqrt{a^2 - x^2} = a \cos \phi$: in the second, $dx = -a \sin \phi\, d\phi$ and $\sqrt{a^2 - x^2} = a \sin \phi$.

b) Differentials involving $\sqrt{a^2 + x^2}$ may sometimes be simplified by putting $x = a \tan \phi$; for, then, $dx = a \sec^2 \phi\, d\phi$ and $\sqrt{a^2 + x^2} = a \sec \phi$.

c) Differentials involving $\sqrt{x^2 - a^2}$ may be simplified by putting $x = a\sec\phi$; for, then, $dx = a\sec\phi\tan\phi\,d\phi$ and $\sqrt{x^2 - a^2} = a\tan\phi$.

d) The form $\sqrt{x^2 + bx + c}$ can be written:

$$\begin{aligned}\sqrt{[x^2 + bx + c]} &\equiv \tfrac{1}{2}\sqrt{[(2x+b)^2 + (4c - b^2)]}\\ &\equiv \tfrac{1}{2}\sqrt{[(2x+b)^2 - (b^2 - 4c)]}.\end{aligned}$$

Now put $z = 2x + b$, and these become:

$$\begin{aligned}\sqrt{[x^2 + bx + c]} &\equiv \tfrac{1}{2}\sqrt{[z^2 + (4c - b^2)]}\\ &\equiv \tfrac{1}{2}\sqrt{[z^2 - (b^2 - 4c)]}.\end{aligned}$$

The first form would be used if $4c > b^2$; the second, if $4c < b^2$.

These may be treated as forms *b*) and *c*) above: the first, by putting $z = \sqrt{4c - b^2}\tan\phi$; the second, by putting

$$z = \sqrt{b^2 - 4c}\sec\phi.$$

e) The form $\sqrt{c + bx - x^2}$ can be written

$$\sqrt{[c + bx - x^2]} = \tfrac{1}{2}\sqrt{[(b^2 + 4c) - (2x - b)^2]}.$$

Put $z = 2x - b$; and this may be treated as *a*), above, by substituting $z = \sqrt{b^2 + 4c}\sin\phi$.

f) The forms $\sqrt{[ax^2 + bx + c]}$ and $\sqrt{[c + bx - ax^2]}$ can be treated as *d*) and *e*), after the factor $\sqrt{a}$ is taken from under the radical sign.

Many important differentials can be so simplified, by one of the foregoing substitutions, that their anti-differentials may be readily found.

29. ANTI-DIFFERENTIATION "BY PARTS."

It has been proved that

$$(1) \qquad duv = udv + vdu. \quad (\S\ 23,\ \text{II.})$$

Transposing this gives

$$(2) \qquad udv = duv - vdu. \quad \text{Whence,}$$

$$(3) \qquad \int udv = uv - \int vdu.$$

Formula (3) is frequently useful in treating a differential which can be broken into two factors, u and dv, such that the anti-differential of dv can be easily found. Some examples of the application of (3) follow.

Example 1. Find $\int \sin^2 x\,dx$.

Put $u = \sin x$, and $dv = \sin x\,dx$; then $du = \cos x\,dx$ and $v = -\cos x$. Hence, by (3),

$$\int \sin^2 x\,dx = -\sin x \cos x + \int \cos^2 x\,dx.$$
$$= -\sin x \cos x + \int (1 - \sin^2 x)\,dx.$$
$$= -\sin x \cos x + \int dx - \int \sin^2 x\,dx.$$

Transposing, etc., and dividing by 2, gives

(4) $$\int \sin^2 x\,dx = \tfrac{1}{2} x - \tfrac{1}{2} \sin x \cos x.$$

[Compare ex. 30, § 23.]

Example 2. Find $\int e^{ax} \sin x\,dx$.

Put $u = \sin x$, and $dv = e^{ax}dx$; then $du = \cos x\,dx$ and $v = \frac{1}{a} e^{ax}$. Then, by (3),

(5) $$\int e^{ax} \sin x\,dx = \frac{1}{a} e^{ax} \sin x - \frac{1}{a} \int e^{ax} \cos x\,dx.$$

In $\int e^{ax} \cos x\,dx$, put $u = \cos x$ and $dv = e^{ax}dx$; we get $du = -\sin x\,dx$, and $v = \frac{1}{a} e^{ax}$: so that

$$\int e^{ax} \cos x\,dx = \frac{1}{a} e^{ax} \cos x + \frac{1}{a} \int e^{ax} \sin x\,dx.$$

Restoring this value in (5) gives

$$\int e^{ax} \sin x\,dx = \frac{1}{a} e^{ax} \sin x - \frac{1}{a^2} e^{ax} \cos x - \frac{1}{a^2} \int e^{ax} \sin x\,dx.$$

Transposing, etc., we get

(6) $$\int e^{ax} \sin x = \frac{e^{ax}(a \sin x - \cos x)}{a^2 + 1}.$$

Verify this result by differentiation.

30. ANTI-DIFFERENTIATION OF CERTAIN TRIGONOMETRIC FORMS.

Find $\int \sin^m x \cos^n x\,dx$.

Put $u = \cos^{n-1} x$, and $dv = \sin^m x \cos x\,dx = \sin^m x\,d(\sin x)$; then $du = -(n-1)\cos^{n-2} x \sin x\,dx$, and $v = \frac{\sin^{m+1} x}{m+1}$. Substituting in (3) of § 29 gives

(1) $$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x\,dx.$$

But, $\sin^{m+2}x = \sin^m x \sin^2 x = \sin^m x(1 - \cos^2 x)$
$= \sin^m x - \sin^m x \cos^2 x.$

Substituting this in (1) gives

$$(2) \qquad \int \sin^m x \cos^n x dx = \frac{\sin^{m+1}x \cos^{n-1}x}{m+1}$$
$$+ \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x dx - \frac{n-1}{m+1}\int \sin^m x \cos^n x dx.$$

Transposing, etc., gives

$$(3) \qquad \int \sin^m x \cos^n x dx = \frac{\sin^{m+1}x \cos^{n-1}x}{m+n}$$
$$+ \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x dx.$$

If we had begun by putting $u = \sin^{m-1}x$ and $dv = \cos^n x \sin x dx$, we should, in a similar way, have obtained

$$(4) \qquad \int \sin^m x \cos^n x dx = -\frac{\sin^{m-1}x \cos^{n+1}x}{m+n}$$
$$+ \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x dx.$$

Formula (1) is valid for all values of m and n except $m = -1$; (3) and (4) are valid for all values of m and n except when $m + n = 0$.

In (1), if we put $m = -n$, we get

$$(5) \qquad \int \operatorname{ctn}^n x dx = -\frac{\operatorname{ctn}^{n-1}x}{n-1} - \int \operatorname{ctn}^{n-2} x dx.$$

Put $n = -m$ in (1) and we get

$$(6) \qquad \int \tan^m x dx = \frac{\tan^{m+1}x}{m+1} - \int \tan^{m+2} x dx.$$

Transposing (6) gives

$$(7) \qquad \int \tan^{m+2} x dx = \frac{\tan^{m+1}x}{m+1} - \int \tan^m x dx.$$

In (7), change $m + 2$ to n, in order to get a more convenient form, and we have

$$(8) \qquad \int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2} x dx.$$

Formulæ (5) and (8) will enable us to find the anti-differential of any power of $\operatorname{ctn} x$, or $\tan x$, whose exponent is a positive integer.

In (3), put $m = 0$, and we get

$$(9) \qquad \int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

In (4), put $n = 0$, and we get

$$(10) \qquad \int \sin^m x\,dx = -\frac{1}{m}\sin^{m-1}x\cos x + \frac{m-1}{m}\int \sin^{m-2}x\,dx.$$

Formulæ (9) and (10) will furnish the anti-differential of any positive integral power of the cosine and sine. If a negative integral power of the cosine or sine is given, it may be handled by one of the following, which are obtained from (9) and (10) by transposing, etc.:

$$(11) \qquad \int \cos^{n-2}x\,dx = -\frac{\sin x \cos^{n-1}x}{n-1} + \frac{n}{n-1}\int \cos^n x\,dx;$$

$$(12) \qquad \int \sin^{m-2}x\,dx = \frac{\sin^{m-1}x\cos x}{m-1} + \frac{m}{m-1}\int \sin^m x\,dx.$$

Exercises.

1. $\int \frac{dx}{\sqrt{2ax - x^2}} = \text{vers}^{-1}\frac{x}{a} = \cos^{-1}\frac{a-x}{a}.$

2. $\int \sqrt{\frac{a+x}{a-x}}\,dx \equiv \int \frac{a+x}{\sqrt{a^2 - x^2}}\,dx = a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}.$

3. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a}\log\frac{x}{a + \sqrt{a^2 \pm x^2}}.$

4. $\int \frac{\sqrt{x^2+1}}{x}\,dx \equiv \int \frac{(x^2+1)\,dx}{x\sqrt{1+x^2}} = \sqrt{1+x^2} + \log\frac{x}{1+\sqrt{1+x^2}}.$

5. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{2}\sin x\cos x.$

6. $\int x\sin x\,dx = \sin x - x\cos x.$

7. $\int x\cos x\,dx = \cos x + x\sin x.$

8. $\int x^2\sin x\,dx = 2x\sin x + (2 - x^2)\cos x,$

9. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}.$

10. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}.$

11. $\int \tan^{-1} x\,dx = x \tan^{-1} x - \frac{1}{2} \log (1 + x^2)$.

12. $\int \operatorname{ctn}^{-1} x\,dx = x \operatorname{ctn}^{-1} x + \frac{1}{2} \log (1 + x^2)$.

13. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = \frac{-x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$.

14. $\int \frac{x^2\,dx}{\sqrt{a^2 + x^2}} = \frac{x\sqrt{a^2 + x^2}}{2} - \frac{a^2}{2} \log (x + \sqrt{a^2 + x^2})$.

15. $\int \sin^3 x\,dx = -\frac{\cos x}{3} (\sin^2 x + 2)$.

16. $\int \cos^3 x\,dx = \frac{\sin x}{3} (\cos^2 x + 2)$.

17. $\int \sin^4 x\,dx = -\frac{\cos x}{4} (\sin^3 x + \frac{3}{2} \sin x) + \frac{3x}{8}$.

18. $\int \tan^3 x\,dx = \frac{1}{2} \tan^2 x + \log \cos x$.

19. $\int \tan^4 x\,dx = \frac{1}{3} \tan^3 x - \tan x + x$.

20. $\int \sin^4 x \cos^3 x\,dx = \frac{\sin^5 x}{7} (\cos^2 x + \frac{2}{5})$.

21. $\int x^n e^{ax}\,dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\,dx$.

22. $\int e^{ax} \cos x\,dx = \frac{e^{ax} (a \cos x + \sin x)}{a^2 + 1}$.

CHAPTER XII.

AREAS BY SUMMATION.

31. AREAS: SUMMATIONS.

Let the curve AB, in fig. 13, be the locus of the function $y = f(x)$; which is finite, positive, continuous, single-valued, and increasing, as x increases from $OM_0 = x_0$ to $OM_n = x_n$. It is required to define, and calculate, the **area** included by the curve, the x-axis, and the ordinates M_0P_0, M_nP_n: or, briefly, *the area under the curve* P_0P_n.

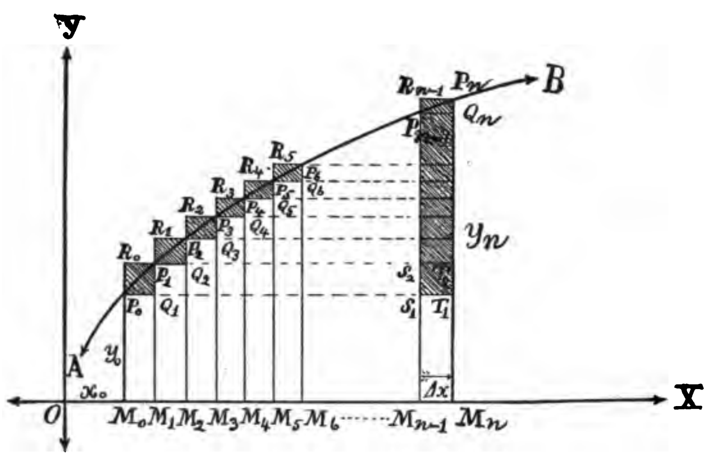

Fig. 13.

Divide the base M_0M_n into n equal parts, n being a positive integer. On these parts as bases, construct the two series of rectangles, the one inscribed in, and the other overlapping, the figure $M_0M_nP_nP_0$; as shown. Represent by V the sum of the overlapping rectangles, and by U the sum of the inscribed rectangles. The difference $V - U$ is, obviously, equal to the sum of the small rectangles Q_1R_0, Q_2R_1, ... Q_nR_{n-1}: and this sum is equal to the single

rectangle T_1R_{n-1}; since all the bases P_0Q_1, P_1Q_2, ... $P_{n-1}Q_n$ are equal, by construction. But, area

$$T_1R_{n-1} = T_1P_n \times S_1T_1 = [M_nP_n - M_0P_0]\,S_1T_1.$$

Hence, since $S_1T_1 = \frac{x_n - x_0}{n}$, we get

$$(1) \qquad V - U = [M_nP_n - M_0P_0]\,\frac{x_n - x_0}{n}.$$

This equation holds good for any integral value of n whatever. Let n be indefinitely increased. For simplicity, suppose n to take, in succession, the set of values $n_k = kn$, where $k = 1, 2, 3, 4, \ldots .^* \infty$. Then, obviously, both V and U are variables; and $V =$ the sum of the overlapping rectangles, is a decreasing variable and is always greater than any value of U; while $U =$ sum of the inscribed rectangles, is an increasing variable and is always less than any value of V.

It follows from (1) that

$$(2) \qquad \lim_{n=\infty}\Big[V - U\Big] = \lim_{n=\infty}\Big[(M_nP_n - M_0P_0)\,\frac{x_n - x_0}{n}\Big].$$

But $M_nP_n - M_0P_0$ is a constant, and $\lim_{n=\infty}\Big[\frac{x_n - x_0}{n}\Big] = 0$; hence,

$$(3) \qquad \lim_{n=\infty}\Big[V - U\Big] = 0.$$

It follows from (3) that, as n is increased indefinitely, *the limits of U and V are identical.* **This limit is the area under the curve** P_0P_n. Denote it by G; then

$$(4) \qquad \lim_{n=\infty}\Big[U\Big] = \lim_{n=\infty}\Big[V\Big] = G.$$

Let us now find an algebraic expression for G.

By construction (see fig. 13)

$$M_0M_1 = M_1M_2 = M_2M_3 = \ldots . = M_{n-1}M_n = \frac{x_n - x_0}{n}.$$

---

* The symbol ∞ means: *the specified variable can take, or is assumed to take, values greater, numerically, than any chosen fixed quantity, A, which may be chosen as great as we please.* The symbol ∞, therefore, does not represent a fixed quantity: *it is the symbol of an endlessly increasing variable.* Such symbolic equations as $z = \infty$ (see ex. 11, p. 15), $n = \infty$, etc., mean, merely, that z, or n, etc., is *increased without limit;* or, as it is expressed, "*indefinitely increased.*"

These are the equal increments of x, as x increases from $OM_0 = x_0$ to $OM_n = x_n$; hence, (§ 22) we may represent their common value by dx.

The abscissæ of the points $P_0, P_1, P_2, \ldots, P_{n-1}$, are $x_0, x_1, x_2, \ldots x_{n-1}$, respectively; and their ordinates are $f(x_0), f(x_1), f(x_2), \ldots f(x_{n-1})$. Now, dx is the common base of the n rectangles composing U; and the ordinates of P_0, P_1, etc., are their altitudes: so that we get

$$(5) \qquad U = f(x_0)\,dx + f(x_1)\,dx + f(x_2)\,dx + \ldots \ldots + f(x_{n-1})\,dx.$$

There are n terms in the right member of equation (5): and if n is indefinitely increased, the factor, $f(x)$, in each term remains finite, since it is the ordinate of some point between P_0 and P_n on the curve in fig. 13; but the factor $dx = \dfrac{x_n - x_0}{n}$ is an infinitesimal. It follows, therefore, that, as n is indefinitely increased, (a) *each term $f(x)\,dx$ is an infinitesimal; and*, (b) *the number of terms increases indefinitely.*

A convenient symbol to represent equation (5) is

$$(6) \qquad U = \sum_{x = x_0}^{x = x_{n-1}} f(x)\,dx,$$

which means: *the sum of the n products obtained by giving x, in $f(x)$, the n values from x_0 to x_{n-1}, having the common difference dx; and then multiplying each result by dx.*

But we have defined the area under P_0P_n as

$$G = \lim_{n=\infty} [U];$$

and, since $\lim\limits_{n=\infty} [x_{n-1}] = OM_n = x_n$, we get from (6)

$$(7) \qquad G = \lim_{n=\infty} \sum_{x = x_0}^{x = x_n} f(x)\,dx.$$

Each of the terms $f(x)\,dx$ being a positive quantity (since $f(x)$ and dx are both positive) the sum U, and the limit G, must be positive, under the conditions stated at the beginning of this section.

We will show (see § 32) that formula (7) is a perfectly general one, for the area included by the continuous curve $y = f(x)$, the x-axis, and the ordinates $f(x_0)$ and $f(x_n)$: and we shall develop a general method, later, by which the limit expressed in (7) can be

computed. A few simple cases may be treated by rather cumbrous algebraic methods.

I. For example, if the straight line, $y = mx$, is given [let the student construct a suitable figure], we get

$$\sum_{x_0}^{x_{n-1}} mx\,dx = mx_0dx + m(x_0 + dx)\,dx + m(x_0 + 2dx)\,dx + \ldots \ldots + m[x_0 + (n-1)\,dx]\,dx$$

$$= mx_0n\,dx + m[1 + 2 + 3 + \ldots + (n-1)]\,(dx)^2$$

$$* = mx_0n\,dx + m\left[\frac{n(n-1)}{2}\right](dx)^2$$

$$= mx_0n\,dx + \frac{m}{2}[(n\,dx)^2 - (n\,dx)\,dx]$$

But, since $\frac{x_n - x_0}{n} = dx$, we get $n\,dx = x_n - x_0$; whence

$$\sum_{x_0}^{x_{n-1}} mx\,dx = mx_0(x_n - x_0) + \frac{m}{2}[(x_n - x_0)^2 - (x_n - x_0)\,dx]$$

$$= (x_n - x_0)\frac{mx_n + mx_0}{2} - \frac{m}{2}(x_n - x_0)\,dx$$

$$= (x_n - x_0)\left[\frac{y_n + y_0}{2} - \frac{m}{2}dx\right].$$

Since $dx \doteq 0$ as n is indefinitely increased, we get

$$G = \lim_{n=\infty}\sum_{x_0}^{x_n} mx\,dx = [x_n - x_0]\left[\frac{y_n + y_0}{2}\right],$$

which is the area of the trapezoid under the line, $y = mx$, from P_0, to P_n, whose base is $M_0M_n = OM_n - OM_0 = x_n - x_0$, and whose parallel sides are $y_0 = mx_0$ and $y_n = mx_n$.

II. Another example which may be worked out algebraically is: *to find the area under the parabola, $y = x^2 + c$, from P_0 to P_n.* (See fig. 14.)

In this case we get

$$\sum_{x_0}^{x_{n-1}} (x^2 + c)\,dx = (x_0^2 + c)\,dx + [(x_0 + dx)^2 + c]\,dx + [(x_0 + 2dx)^2 + c]\,dx + \ldots \ldots + [(x_0 + \{n-1\}\,dx)^2 + c]\,dx$$

$$= (x_0^2 + c)\,n\,dx + 2x_0[1 + 2 + 3 + \ldots + (n-1)]\,(dx)^2 + [1^2 + 2^2 + 3^2 + \ldots + (n-1)^2]\,(dx)^3$$

$$\dagger = \left\{x_0^2 + c + x_0[ndx - dx] + \tfrac{1}{3}(ndx)^2 - \tfrac{1}{2}(ndx)dx + \tfrac{1}{6}(dx)^2\right\}ndx$$

$$= \left\{x_0^2 + c + x_0[(x_n - x_0) - dx] + \tfrac{1}{3}(x_n - x_0)^2 - \tfrac{1}{2}(x_n - x_0)\,dx + \tfrac{1}{6}(dx)^2\right\}(x_n - x_0),$$

since $n\,dx = x_n - x_0$. From this we get, since $\lim_{n=\infty}[dx] = 0$,

---

* $1 + 2 + 3 + 4 + \ldots + (n-1) = \frac{(n-1)\,n}{2}.$

† $1^2 + 2^2 + 3^2 + 4^2 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}.$

$$G = \lim_{n=\infty} \sum_{x_0}^{x_n} (x^2 + c)\,dx = [x_0^2 + c + x_0(x_n - x_0) + \tfrac{1}{3}(x_n - x_0)^2](x_n - x_0)$$
$$= [\tfrac{1}{3}(x_n^2 + x_n x_0 + x_0^2) + c](x_n - x_0)$$
$$= \tfrac{1}{3}(x_n^3 - x_0^3) + c(x_n - x_0).$$

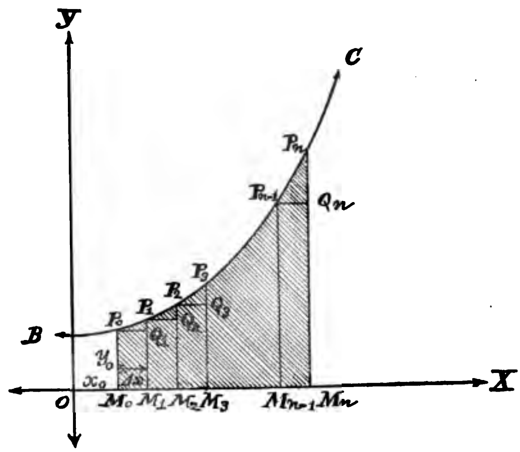

Fig. 14.

Exercises.

1. Given the formulæ

$$e^a + e^{a+\phi} + e^{a+2\phi} + \ldots\ldots + e^{a+(n-1)\phi} = e^a\left(\frac{e^{n\phi} - 1}{e^{\phi} - 1}\right),$$

$$\text{and } \lim_{\phi \doteq 0}\left[\frac{\phi}{e^{\phi} - 1}\right] = 1;$$

show that the area under the curve, $y = e^x$, from $x_0 = a$ to $x_n = b$, is $e^b - e^a$.

2. Given the formula

$\sin a + \sin(a + \phi) + \sin(a + 2\phi) + \ldots\ldots$

$$\ldots\ldots + \sin[a + (n-1)\phi] = \left[\cos\left(a - \frac{\phi}{2}\right) - \cos\left(a + n\phi - \frac{\phi}{2}\right)\right]\frac{\frac{1}{2}}{\sin\frac{1}{2}\phi};$$

show that the area under the curve, $y = \sin x$, from $x_0 = 0$ to $x_n = \dfrac{\pi}{2}$, is 1.

32. AREAS WHEN $y = f(x)$ IS DECREASING: AND OTHER IMPORTANT CASES.

Let the curve (see fig. 15) *descend* from P_0 to P_n. It is obvious, that the base M_0M_n can be divided into equal parts; that, on these parts as bases, can be constructed two series of rectangles, the one inscribed in, and the other overlapping, the area G, under P_0P_n; and that, in short, the definition and formulæ of § 31 will apply equally well to the *decreasing*, positive, function represented by the curve from P_0 to P_n. In this case the area G must be positive, since both $f(x)$ and dx are positive as x increases from $OM_0 = x_0$ to $OM_n = x_n$.

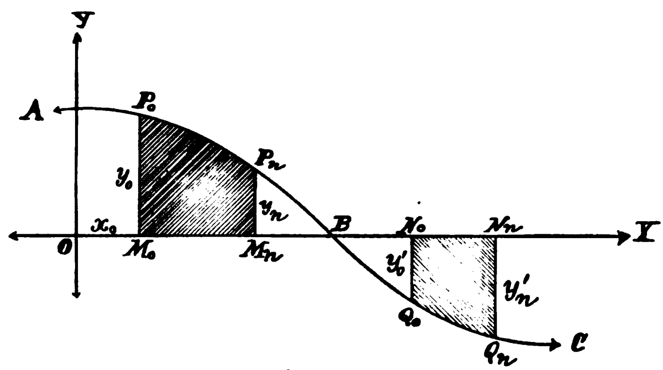

Fig. 15.

In like manner it may be seen that, if the area $G' = N_0Q_0Q_nN_n$ lies *below the x-axis*, we shall get, as in § 31,

$$G' = \lim_{n=\infty} \sum_{x'_0}^{x'_n} f(x)\,dx;$$

where $x'_0 = ON_0$ and $x'_n = ON_n$, and $y = f(x)$ is the equation of the curve Q_0Q_n.

In this case dx is positive, because x is increasing from x'_0 to x'_n; but $f(x)$ is negative, because the curve Q_0Q_n lies below the x-axis. Hence, each term $f(x)\,dx$ is negative, and *G' is negative, when the area is below the x-axis.*

It follows, then, from the foregoing, and § 31, that the formula

$$G = \lim_{n=\infty} \sum_{x_0}^{x_n} f(x)\,dx$$

gives a positive or a negative result, when $x_0 < x_n$, according as the area is above or below the x-axis.

Moreover, it may be seen that, if $x_0 > x_n$, so that x is decreasing and dx is negative, then the above rule for the algebraic sign of G is reversed.

It may be seen, also, that the above results hold good when either x_0 or x_n is negative; or when both of them are negative. (See fig. 16.)

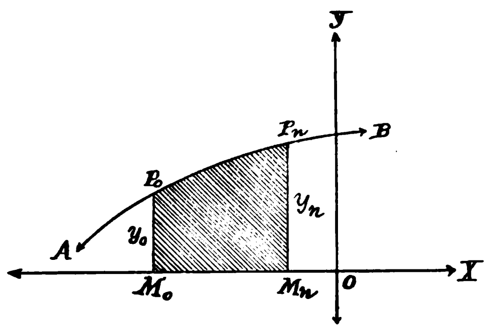

Fig. 16.

Hence, it follows that formula (7) of § 31 is a perfectly general one, for the area between a continuous curve and the x-axis, from x_0 to x_n.

A general method of finding G, based on anti-differentiation, will be developed in the next Chapter.

Exercises.

1. Find the area under $y = \sin x$ from $x_0 = \frac{\pi}{2}$ to $x_n = \pi$; also, from $x_0 = \pi$ to $x_n = \frac{3\pi}{2}$. (See ex. 2, § 31.)

2. Find the area under $y = \sin x$ from $x_0 = -\frac{\pi}{2}$ to $x_n = 0$; also, from $x_0 = -\frac{\pi}{2}$ to $x_n = \frac{\pi}{2}$.

3. Find the area between the line, $y = 2x$, and the x-axis, from $x_0 = -8$ to $x_n = -2$; also, from $x_0 = 2$ to $x_n = 8$; also, from $x_0 = 8$ to $x_n = 2$; also, from $x_0 = -3$ to $x_n = 3$.

4. Find the area between the parabola, $y = x^2 - 9$, and the x-axis.

Ans. -36.

5. Find the area of the segment cut off the parabola, $y = x^2 - 9$, by the chord, $y = 3x - 9$.

CHAPTER XIII.

DIFFERENTIAL OF AREA: A GENERAL METHOD OF SUMMATION: INTEGRATION.

33. THE DIFFERENTIAL OF THE AREA UNDER A FIXED CURVE.

Let the curve AB, in fig. 17, be the locus of the function $y = f(x)$, which is finite, continuous, and single-valued, for all the values of x considered in the following discussion. Let $OM_0 = x_0$, and $OM = x$, be the abscissæ of the points P_0 and P. Then, by equation (7), § 31, the area under P_0P is

$$(a) \qquad G = \lim_{n=\infty} \sum_{x_0}^{x} f(x)\,dx,$$

where $n\,dx = x - x_0$, and $x > x_0$.

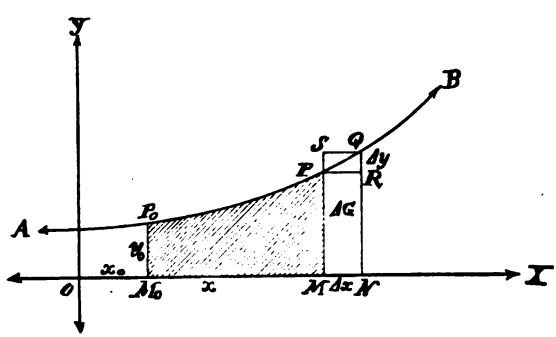

Fig. 17.

The following deductions are obvious: (a) OM_0, and the curve AB, both being fixed, G will vary with OM, and will be fixed if OM is fixed; hence G *must be some function of the terminal abscissa* OM: (b) similarly, G must be some function of the *initial* abscissa OM_0: and (c) if OM_0 and OM are both fixed, G may have different values for different functions, — that is, for different forms of the function $f(x)$; since this will cause the curve P_0P to change posi-

tion, — hence G is closely dependent on the form of $f(x)$. This dependence is stated in the following:

Theorem. *The differential of the area, G, under the curve $y = f(x)$, measured from the fixed ordinate M_0P_0, to the variable ordinate MP, is $f(x)\,dx$.*

Proof. Let OM_0, and the locus AB (fig. 17), of $y = f(x)$, be fixed. Give to x the increment $MN = \Delta x$: then x will take the new value, $x = OM + MN$; and G will take the new value, $M_0MPP_0 + MNQP$. The increment of G is $MNQP$, and we may call it ΔG, or

$$(1) \qquad \Delta G = MNQP.$$

Let PR and SQ be drawn parallel to OX: then, $MP = f(x) = NR$; and $NQ = NR + RQ = f(x) + \Delta f(x)$.

Now, area $MNRP = MP \times MN$; and area $MNQS = NQ \times MN$; hence,

$$(2) \qquad MNRP = f(x)\,\Delta x, \text{ and}$$

$$(3) \qquad MNQS = [f(x) + \Delta f(x)]\,\Delta x.$$

It is obviously true that,

$$*(4) \qquad \text{area } MNRP < \text{area } MNQP < \text{area } MNQS.$$

$$\therefore\ (5) \qquad f(x)\,\Delta x < \Delta G < [f(x) + \Delta f(x)]\,\Delta x.$$

$$\therefore\ (6) \qquad f(x) < \frac{\Delta G}{\Delta x} < f(x) + \Delta f(x).$$

But $\lim_{\Delta x \doteq 0} [\Delta f(x)] = 0$, since $f(x)$ is continuous by hypothesis. (See § 14.) It follows, therefore, from (6) that

$$(7) \quad f(x) = \lim_{\Delta x \doteq 0} \left[\frac{\Delta G}{\Delta x}\right] = D_x G = \frac{dG}{dx}. \quad [\S\S\, 9, 22, eq. (6).]$$

We get, then, from (7)

$$(8) \qquad dG = f(x)\,dx = y\,dx, \qquad q.\ e.\ d.$$

It has been noted already that *the area, G, from the fixed ordinate M_0P_0, must be some function of the terminal abscissa $OM = x$.* The demonstration given above shows that *this unknown function of x, is the anti-differential of $f(x)\,dx$*, where $y = f(x)$ is the equation of the given curve.

* If $f(x)$ is decreasing, the inequalities, (4), (5), and (6), are reversed.

We get, therefore, from equation (8)

$$(9) \qquad G = \int f(x)\,dx = \phi(x) + K, \text{ [by (16), § 24]}.$$

Substituting in (a), above, we get

$$(10) \qquad G = \lim_{n=\infty} \sum_{x_0}^{x} f(x)\,dx = \int f(x)\,dx = \phi(x) + K.$$

Equation (10) suggests a more convenient symbol than

$$\lim_{n=\infty} \sum_{x_0}^{x} f(x)\,dx,$$

to represent this infinite summation [see equation (6), § 31]. *The new symbol is* $\int_{x_0}^{x}$; which is a modification of the symbol of anti-differentiation; by affixing, as shown, the initial and terminal abscissæ of the area to be computed. Let us call this new symbol, $\int_{x_0}^{x}$, **the sign of integration.** It is, as explained in § 31 and above, a **symbol of summation;** or, in fact, a symbol expressing the **limit of an infinite summation.**

Moreover, this summation, or integration, of the differential terms $f(x)\,dx$, is performed for values of x extending consecutively, or continuously, from $x_0 = OM_0$ to $x = OM$. These values of x are called **the limits of integration;** the one placed beneath the symbol $\int$ being called the * *the lower limit of x;* and the one at the top being called the * *upper limit of x.* The variable x is supposed, *always*, to change value *from* the lower limit *to* the upper; hence, *x is increasing, and dx is positive, when the upper limit is greater than the lower ;* and, *x is decreasing, and dx is negative, when the upper limit is less than the lower.*

The symbol $\int$ will still be used to express the operation of anti-differentiation, as explained in § 24.

* The phrases "lower limit of x," and "upper limit of x," are equivalent to "initial value of x" and "terminal value of x," respectively.

We shall, therefore, express the area $G = M_0MPP_0$ (see fig. 17):

$$(11)\qquad G = \int_{x_0}^{x} f(x)\,dx = \int_{x_0}^{x} y\,dx = \phi(x) + K.$$

The arbitrary constant K is easily determined; for, if $x = x_0$, MP will coincide with M_0P_0 and the area is zero. Substituting in (11) we get

$$(12)\qquad 0 = \int_{x_0}^{x_0} f(x)\,dx = \phi(x_0) + K;$$

whence, $K = -\phi(x_0)$; and (11) becomes

$$(13)\qquad G = \int_{x_0}^{x} f(x)\,dx = \phi(x) - \phi(x_0).$$

If we take a fixed value x_n for the upper limit of integration, then the ordinate MP (fig. 17) is fixed, and the area G is fixed. In this case (13) becomes

$$(14)\qquad G = \int_{x_0}^{x_n} f(x)\,dx = \phi(x_n) - \phi(x_0).$$

A compact notation for the difference in the right member of (14) is

$$(15)\qquad \phi(x_n) - \phi(x_0) \equiv \phi(x)\Big|_{x_0}^{x_n}.$$

The symbol $\phi(x)\Big|_{x_0}^{x_n}$ expresses:—*Substitute for x in $\phi(x)$, first, the value x_n, then, the value x_0; and, finally, subtract the second result from the first.*

We get, now, the final symbolic representation of the area between the curve $y = f(x)$ and the x-axis from x_0 to x_n,

$$(16)\qquad G = \int_{x_0}^{x_n} y\,dx \equiv \int_{x_0}^{x_n} f(x)\,dx = \phi(x)\Big|_{x_0}^{x_n} \equiv \phi(x_n) - \phi(x_0),$$

in which $\phi(x) = \int f(x)\,dx$.

The student will not realize, speedily, the wonderful range and power of the instrument he has fashioned in formula (16). He should carefully review all the work leading up to it, in this and in preceding chapters; and satisfy himself, step by step, of the soundness of the argument upon which it rests. The applications of the formula are multitudinous; being in no wise limited to the mere problem of calculating areas. We have, in truth, merely utilized the area under a curve, as a means of making as clear and concrete as possible, the deduction of this very important general formula. Its superiority over any method of calculation hitherto known to the student, even for finding areas, may quickly be discovered by calculating, now, by formula (16), the areas given in the exercises at the end of §§ 31 and 32.

For example, the area under the straight line $y = mx$, from x_0 to x_n is

$$G = \int_{x_0}^{x_n} mx\,dx = \frac{mx^2}{2}\Bigg|_{x_0}^{x_n} = \frac{mx_n^2}{2} - \frac{mx_0^2}{2} \equiv (x_n - x_0)\,\frac{mx_n + mx_0}{2}.$$

Again, the area under the parabola $y = x^2 + c$, from x_0 to x_n, is

$$G = \int_{x_0}^{x_n} (x^2 + c)\,dx = \left(\frac{x^3}{3} + cx\right)\Bigg|_{x_0}^{x_n} = \frac{1}{3}(x_n^3 - x_0^3) + c(x_n - x_0).$$

Exercise.

1. Solve again, using formula (16), the exercises at the end of §§ 31 and 32.

34. REMARKS ON INTEGRATION.

The **process of summation** expressed in formula (16), § 33, we shall call **integration.** It is an operation which furnishes as result, the **limit of the sum** of the infinite number of differential (or infinitesimal) terms, $f(x)\,dx$, which may be formed between the initial value x_0 and the terminal value x_n, as explained in (5), (6), (7), § 31.

The result, G, in eq. (16), § 33, is called **the integral of** $f(x)\,dx$ **from** x_0 **to** x_n; which means, *the limit of the sum of the infinitesimal terms* $f(x)\,dx$ *from* x_0 *to* x_n.

This expression is known, also, as *the integral of* $f(x)\,dx$ *between the limits* x_0 *and* x_n.

What we have defined, in § 24, as *the anti-differential* of $f(x)\,dx$, is what is commonly called *the indefinite integral* of $f(x)\,dx$; and, a

table of *anti-differentials* is usually called a table of *integrals.* [See B. O. Peirce's *Short Table of Integrals*, published by Ginn & Co.]

According to common practice, the operation symbolized by $\int_{x_0}^{x_n}$ (which we have called *integration*) is called *definite integration;* and the result is called a *definite integral.*

There would be a distinct gain in simplicity if the term *indefinite integral* were no longer used; and *anti-differential* were used instead. The objectionable words "indefinite" and "definite," before integral, could then be dropped altogether; and the true nature of the "indefinite integral" would be suggested by the name "anti-differential." Then, the word "integral," without the confusing adjective "definite," could be used to stand for G, as symbolized in formula (16) § 33. "*To integrate*" should mean, only:—*Perform the operations of formula* (16), § 33; and not, also:—*Find the anti-differential of*, etc. The main difficulty appears to be in the impracticable (?) words *anti-differentiate*, and *anti-differentiation*, which would replace the words *integrate*, and *integration*, when the latter are applied to the process of finding *indefinite integrals.*

For the present, however, custom compels us to understand the equivalent phrases:—"*Find the integral of the given differential,*" and "*Integrate the given differential,*" as expressing:—"*Find the anti-differential of the given differential,*" *in addition to the meaning we have stated for these phrases in* § 33; since the symbol, $\int$, is described, generally, as "*the sign of integration*"; this phrase not being restricted, as we would wish it, to describe the symbol $\int_{x_0}^{x_n}$.

Exercises.

Find the values of the following integrals:

1. $\int_0^3 (2x + 3x^2)\,dx = 36;$

2. $\int_0^a \frac{dx}{a^2 + x^2} = \frac{\pi}{4a};$

3. $\int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{2}\log 2;$

4. $\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{1}{2}\log 2;$

5. $\int_1^4 \frac{x\,dx}{\sqrt{2 + 4x}} = \frac{3\sqrt{2}}{2}$ [SUGGESTION: Put $2 + 4x = z^2$];

6. $\int_0^2 e^{ax}\,dx = \frac{e^{2a} - 1}{a};$

7. $\int_1^e x^2 \log x\,dx = \frac{2e^3 + 1}{9};$

8. $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4};$

9. $\int_0^1 (1 - x^2)^n\,dx = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6 \ldots\ldots\ldots 2n}{3\cdot 5\cdot 7 \ldots\ldots (2n+1)};$

10. $\int_0^1 (1 - x^2)^{n-\frac{1}{2}}\,dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)}{2\cdot 4\cdot 6 \ldots\ldots\ldots 2n}\,\frac{\pi}{2};$

11. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab};$

12. $\int_0^1 \frac{dx}{1 + 2x\cos k + x^2} = \frac{k}{2\sin k};$

13. $\int_0^\pi \frac{x \sin x\,dx}{1 + \cos^2 x} = \frac{\pi^2}{4}.$

CHAPTER XIV.

AREAS BY INTEGRATION: THEOREMS ON INTEGRALS: CERTAIN SPECIAL CASES OF AREAS.

35. APPLICATIONS TO AREAS.

We have now developed a method of obtaining the area of a curve, when its equation is given in rectangular coördinates, by a process of integration, or summation. The datum required is the differential of the area, — called the **differential element.** To completely solve the problem we should:

a) know the position with respect to the axes, and the shape, approximately at least, of the locus of $y = f(x)$ [$f(x)\,dx$ being the given differential element];

b) know, also, the initial and final values of the variable, which are the limits of integration; and

c) find the anti-differential (or "indefinite integral") of the given differential.

The remainder of the work, ordinarily, consists of simple substitutions and reductions.

Of course, the student must examine whether the function (or curve), $y = f(x)$, satisfies all the initial conditions of being finite, continuous, and single-valued (§ 33, at beginning); not only for all values of x *between the limits* of integration, but also *at the limits* themselves. He must discover, also, whether the curve, $y = f(x)$, crosses the x-axis between the limits of integration; so as to determine whether the area sought lies wholly on one side of the x-axis, or not (§ 32).

It has been implied, further, that the limits of integration are both finite; also, that all the quantities involved should be real, and not imaginary.

Certain important exceptions to some of the foregoing restrictions will be considered later. (See §§ 38, 39.)

The student will find, in § 40, certain cases of integrals treated, whose anti-differentials have more than one value for a given value of x. For example,

$$\int_{-1}^{1}\frac{dx}{1+x^2}=\tan^{-1}x\Big|_{-1}^{1}, \quad \text{and} \quad \int_a^b\frac{dx}{\sqrt{c^2-x^2}}=\sin^{-1}\frac{x}{c}\Big|_a^b.$$

Exercises.

1. Find the area of an arch (*a*) of the curve, $y=\sin x$; (*b*) of the curve, $y=\cos x$.

2. Find the area under the following curves between the limits given:

(*a*) $y=3x^2$, from $x=1$ to $x=3$;
(*b*) $y=\frac{1}{x}$, " $x=e$ " $x=e^2$;
(*c*) $y=x^m$, " $x=0$ " $x=1$;
(*d*) $xy^2=1$, " $x=1$ " $x=4$;
(*e*) $y=e^x$, " $x=0$ " $x=2$.

3. Find the area above the x-axis, of the curve, $y=6x-x^2-5$; also, find the area of the segment cut off this curve by the line $y=2x-2$.

4. Find the area cut off the parabola, $y^2=4px$, by the chord $x=a$.

5. Find the areas of the circles (*a*) $x^2+y^2=a^2$; (*b*) $x^2+y^2=2ax$.

6. Find the area of the segment of the ellipse, $9y^2+4x^2=36$, included between the parallel lines, $x=1$ and $x=2$. *Ans.* 3.4346.

7. Find the area of the ellipse, $b^2x^2+a^2y^2=a^2b^2$.

8. Show that the area under the hyperbola, $xy=1$, from $x=1$ to $x=x_1$, is $\log x_1$.

9. Find the area of the segment cut off the hyperbola, $b^2x^2-a^2y^2=a^2b^2$, by the line, $x=a\sqrt{2}$.

10. Find the area under the catenary, $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$, from $x=0$ to $x=2a$.

11. Find the area of one arch of the cycloid, $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$. *Ans.* $3\pi a^2$.

SUGGESTION: *Express $y\,dx$ in terms of θ, and find the initial and final values of θ for limits of integration.*

12. Find the area of the hypocycloid, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

13. The equation of the Lemniscate is $(x^2+y^2)^2=a^2x^2$; find its total area. *Ans.* $\frac{\pi a^2}{2}$.

SUGGESTION: $ax-x^2\equiv\frac{a^2}{4}-\left(x-\frac{a}{2}\right)^2$.

14. Show that the area included between two curves, $y = f(x)$ and $y = \phi(x)$, from x_0 to x_n, is $G = \int_{x_0}^{x_n} [f(x) - \phi(x)]\,dx$, when $f(x) > \phi(x)$.

15. Find the area of the circle, $(x-a)^2 + (y-b)^2 = r^2$.

16. Find the area cut off the catenary, $y = \frac{1}{2}(e^x + e^{-x})$, by the line, $y = \frac{1}{2}(e + e^{-1})$. [Take $e = 2.72$.] *Ans.* 0.74.

17. If oblique coördinate axes, whose included angle is ω, are used, show that the formula for area under a given curve, $y = f(x)$, is $G = \sin\omega \int_{x_0}^{x_n} y\,dx$.

36. AREA BETWEEN A GIVEN CURVE AND THE y-AXIS.

Let the curve AB, in fig. 18, be the locus of the equation, $\phi(x, y) = 0$: and let it be required to find the area $N_0N_nP_nP_0$, between the curve and the y-axis, from $y_0 = ON_0$ to $y_n = ON_n$.

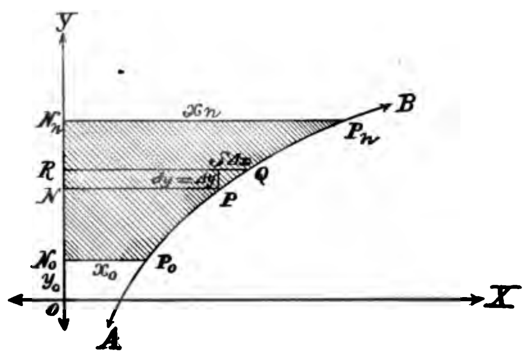

Fig. 18.

In this case, we should regard y as independent variable (§ 2), and, if possible, solve $\phi(x, y) = 0$ for x; so as to get it into the form,

(1) $x = f(y)$.

Now, divide $N_0N_n = y_n - y_0$ into n equal parts, of which

$$NR = \frac{y_n - y_0}{n} = dy^*$$

---

* The student should note that $dy = \Delta y$, when y is independent variable (§ 22). In such case x is the function, and $dx \lesseqgtr \Delta x$.

is one; and construct upon them, as bases, the series of n inscribed rectangles, of which $NRSP = NP \times NR = f(y)\,dy$ is one. Then, as in § 31, we get

$$(2) \qquad \text{area } N_0N_nP_nP_0 = \lim_{n=\infty} \sum_{y_0}^{y_n} f(y)\,dy;$$

and (§ 33) the value of this limit is $\int_{y_0}^{y_n} f(y)\,dy$: whence we get

$$(3) \qquad \text{area } N_0N_nP_nP_0 = \int_{y_0}^{y_n} f(y)\,dy = \int_{y_0}^{y_n} x\,dy.$$

Exercises.

1. Find the area between the parabola, $y^2 = 4px$, the y-axis, and the line, $y = c$.

2. Find the areas of the circle, $x^2 + y^2 = a^2$; and of the ellipse, $b^2x^2 + a^2y^2 = a^2b^2$, by using (3).

3. Find the area between the hyperbola, $xy = c$, the y-axis, and the lines, $y_0 = a$, and $y_n = b$.

4. Find the area included by the parabola, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, and the coördinate axes.

5. Find the area between the semi-cubical parabola, $ay^2 = x^3$, the y-axis, and the line, $y = b$.

6. Find the area between the y-axis, and that arc of the involute of the circle which is traced while ϕ varies from 0 to $\frac{\pi}{2}$. The equation of the involute is

$$\left\{ \begin{array}{l} x = a(\cos\phi + \phi\sin\phi) \\ y = a(\sin\phi - \phi\cos\phi) \end{array} \right\}.$$

Ans. $\frac{\pi a^2}{4}\left[\frac{\pi^2}{12} + 1\right]$.

SUGGESTION: *Express $x\,dy$ in terms of ϕ.*

37. TWO IMPORTANT GENERAL THEOREMS ON INTEGRALS.

A. *If the limits of integration, x_0 and x_n, are interchanged the sign, alone, of the integral is changed.*

Let the given integral be

$$(1) \qquad \int_{x_0}^{x_n} f(x)\,dx = \phi(x)\Big|_{x_0}^{x_n} = \phi(x_n) - \phi(x_0).$$

Interchanging the limits gives

$$(2) \qquad \int_{x_n}^{x_0} f(x)\,dx = \phi(x)\Big|_{x_n}^{x_0} = \phi(x_0) - \phi(x_n).$$

But, $\phi(x_n) - \phi(x_0) = -[\phi(x_0) - \phi(x_n)]$; whence, the theorem,

$$(3) \qquad \int_{x_0}^{x_n} f(x)\,dx = -\int_{x_n}^{x_0} f(x)\,dx.$$

Exercise.

Find the area of the arch, OAB (fig. 2), of the curve, $y = x^3 - 6x^2 + 9x$; and compare the results obtained by integrating, first, from O to B, then, from B to O.

B. *Any given integral can be broken up into as many parts as we please, by breaking up the interval x_0 to x_n into consecutive parts;* that is, if (fig. 19) $M_0M_n = M_0M_1 + M_1M_2 + M_2M_n$, then

$$(1) \qquad \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_n} f(x)\,dx.$$

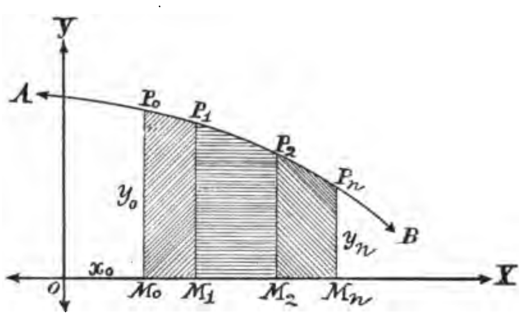

Fig. 19.

Proof:—If $y = f(x)$ is the equation of AB; then equation (1) is equivalent to $M_0M_nP_nP_0 = M_0M_1P_1P_0 + M_1M_2P_2P_1 + M_2M_nP_nP_2$, which is obviously true; hence equation (1), or the theorem, is true.

Theorem B enables us to calculate areas which must be broken up into parts. This is necessary sometimes from the fact (*a*) that the

upper boundary is composed of portions of different curves, or (*b*) that the curve crosses the x-axis one or more times between the limits of integration. In the latter case, the algebraic signs of the parts must be regarded.

Exercises.

1. Find the area of the triangle whose sides are (*a*) $y = 0$, (*b*) $3y = 2x$, and (*c*) $x + y = 5$. Solve, first, by integration; then, by any other method.

2. Find, by integration, the area of the quadrilateral whose sides are (*a*) $y = 0$, (*b*) $y = x$, (*c*) $4y = x + 6$, (*d*) $x + y = 9$.

3. Find the area common to the circle, $x^2 + y^2 = 12x$, and the parabola, $y^2 = 6x$.
Ans. $18\pi + 48$.

4. Find the entire area between the x-axis and the curve,
$y = \frac{1}{5}(x + 2)(x - 2)(x - 5)$, from $x = -3$ to $x = 6$.

38. SOME CASES WHEN $f(x) = \infty$ AT, OR BETWEEN, THE LIMITS OF INTEGRATION.

A. The locus of the function $y = \dfrac{1}{\sqrt{1 - x^2}}$, is the curve BAC, in fig. 20, — the branches AB and AC approaching the asymptotes DF and EG, since $y = \infty$ if $x = \mp 1$. The curve is symmetrical

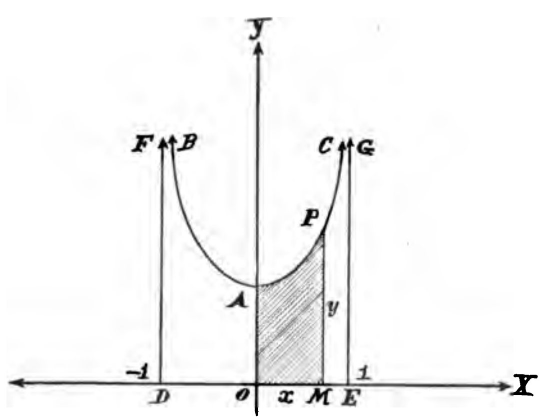

Fig. 20.

with respect to the y-axis, since the equation contains no odd power of x; hence, the entire area under the curve, between the asymptotes,

must be double the area under the branch AC. Now, the branch AC extends upwards without limit; hence, it may happen that the area under the arc AP increases indefinitely as $MP \doteq EG$. This can, frequently, be determined from the form of the anti-differential. We should get, for the area $OMPA$, under AP,

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \Big|_0^x = \sin^{-1}x.$$

The $\sin^{-1}x$ approaches $\frac{\pi}{2}$ as $x \doteq 1$; hence, we get, for the limit of $OMPA$ as $x \doteq 1$,

$$(1) \qquad \text{area } OACE = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x \Big|_0^1 = \frac{\pi}{2}.$$

It follows that the entire area under the curve BAC, from $x = -1$ to $x = 1$, is π.

B. The locus of the function $y = \frac{x}{(1-x)^{\frac{1}{3}}}$, is discontinuous at $x = 1$. From $x = 0$ to $x = 1$, the curve rises from O (fig. 21) to the asymptote CB, which it touches at infinity. From $x = 1$ to $x = 2$, the curve rises from the negative end, at minus infinity, of its asymptote CD. These results may be seen by letting x approach 1, first, from a value less than 1; then, from a value greater than 1.

The area under OP is

$$(2) \qquad OMP = \int_0^x \frac{x\,dx}{(1-x)^{\frac{1}{3}}} = \left[\frac{3}{5}(1-x)^{\frac{5}{3}} - \frac{3}{2}(1-x)^{\frac{2}{3}}\right]\Big|_0^x$$

$$= \frac{3}{5}(1-x)^{\frac{5}{3}} - \frac{3}{2}(1-x)^{\frac{2}{3}} + \frac{9}{10}.$$

Now, if $x \doteq 1$, this result shows that the area OMP approaches $\frac{9}{10}$; hence, the area under the infinite branch OA, is $\frac{9}{10}$.

Again, let $1 < ON < OR$: the area $NQSR$ is

$$(3) \qquad \int_{ON}^{OR} \frac{x\,dx}{(1-x)^{\frac{1}{3}}} = \left[\frac{3}{5}(1-x)^{\frac{5}{3}} - \frac{3}{2}(1-x)^{\frac{2}{3}}\right]\Big|_{ON}^{OR}.$$

Put $OR = 2$, and $ON = x > 1$; we shall get, from (3),

$$(4) \qquad \text{area } NQSR\Big|_x^2 = -\frac{21}{10} - \left[\frac{3}{5}(1-x)^{\frac{5}{3}} - \frac{3}{2}(1-x)^{\frac{2}{3}}\right].$$

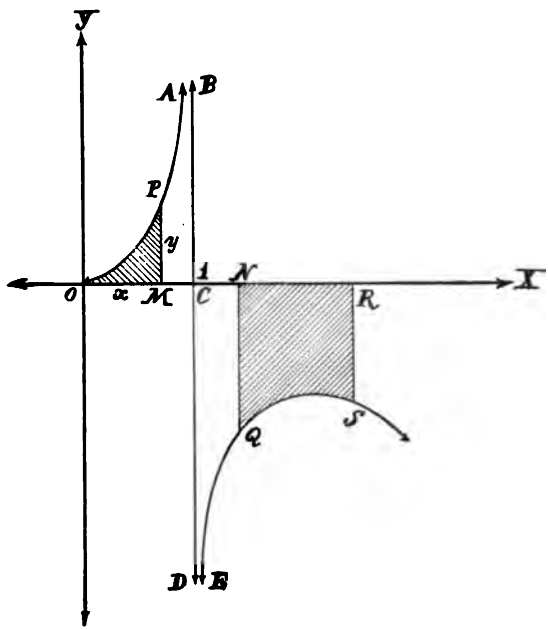

Fig. 21.

This result approaches $-\frac{21}{10}$ as $x \doteq 1$; hence, the area between the curve EQS and the x-axis, measured from the asymptote CD, is finite and determinate if $OR = 2$.

Each case in which $f(x) = \infty$ at, or between, the limits of integration, must be investigated to determine whether, or not, the integral is finite and determinate.

Exercises.

1. Verify the following results:

$a) \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{\pi}{4};$ $\quad b) \int_0^1 \frac{x^4\,dx}{\sqrt{1-x^2}} = \frac{3\pi}{16};$ $\quad c) \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = 1;$

$d) \int_0^1 \frac{x^3\,dx}{\sqrt{1-x^2}} = \frac{2}{3};$ $\quad e) \int_0^1 \frac{dx}{1-x^2} = \infty.$

2. Find the area between the Cissoid, $y^2(2a - x) = x^3$, and its asymptote, $x = 2a$. *Ans.* $3\pi a^2$.

3. Show that the area between the Witch, $xy^2 = 4a^2(2a - x)$, and its asymptote, is $4\pi a^2$.

39. ONE, OR BOTH, OF THE LIMITS OF INTEGRATION MAY BE INFINITE.

A. The locus of the function $y = \frac{1}{1+x^2}$, when $x > 0$, is the curve BA (fig. 22), which approaches the x-axis as an asymptote. The area $OMPB$ under the curve, from the y-axis to MP, is

$$\int_0^x \frac{dx}{1+x^2} = \tan^{-1}x \Big|_0^x = \tan^{-1}x.$$

Now, if x increases without limit, the area, $OMPB = \tan^{-1}x$, will approach the limit $\frac{\pi}{2}$; hence, the area under the entire arc BA, to infinity, is $\frac{\pi}{2}$.

If x is decreased from 0 to $-\infty$, a branch extending from B to the left, indefinitely, will be traced as the remainder of the locus of $y = \frac{1}{1+x^2}$. The two parts, — from $x = -\infty$ to $x = 0$, and from $x = 0$ to $x = \infty$, — are symmetrical with respect to the y-axis; so that the areas on the right and left of the y-axis are equal.

It is evident, therefore, that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-\infty}^{\infty} = 2 \times \text{area under } BA = \pi.$$

In general, whenever a given integral,

$$\int_a^z f(x)\,dx = \phi(x)\Big|_a^z = \phi(z) - \phi(a),$$

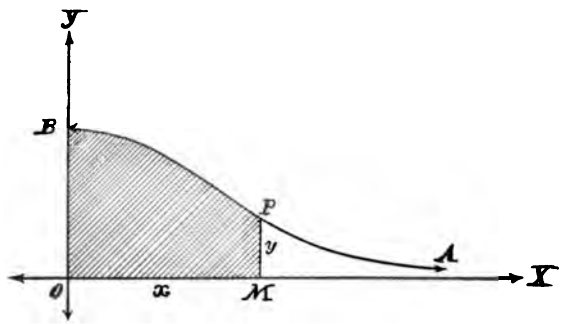

Fig. 22.

is such that $\lim\limits_{z=\infty}\,[\phi(z)]$, is finite and determinate, then $\int_a^\infty f(x)\,dx$ is finite and determinate: and, in like manner the case may be treated, when the lower limit of integration decreases to minus infinity.

Exercises.

1. Prove the following:

$a)\ \int_0^\infty x e^{-ax}\,dx = \frac{1}{a^2};$ $\qquad b)\ \int_0^\infty \frac{dx}{a+bx^2} = \frac{\pi}{2\sqrt{ab}};$

$c)\ \int_0^\infty \frac{dx}{1+2x\cos k + x^2} = \frac{k}{\sin k}.$

2. The equation of any hyperbola, when referred to its asymptotes as axes, is $xy = k$. Show that the area between one branch of the hyperbola and its asymptotes, is infinite.

CHAPTER XV.

MANY-VALUED ANTI-DIFFERENTIALS: VOLUMES OF SOLIDS OF ROTATION.

40. ON CERTAIN INTEGRALS IN WHICH THE ANTI-DIFFERENTIAL IS MANY-VALUED.

Problem I. *Let it be required to find the area under the curve ABC, in fig. 23, from $x_0 = -1$ to $x_n = 1$; the equation of the curve being* $y = \dfrac{1}{1+x^2}$.

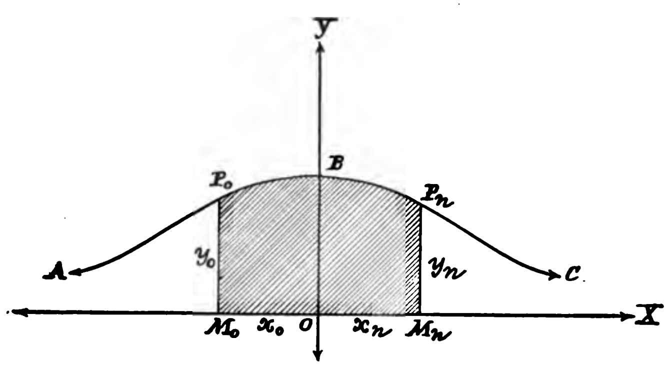

Fig. 23.

The differential element of area is $y\,dx = \dfrac{dx}{1+x^2}$; and its anti-differential is

$$(1)\qquad \int \frac{dx}{1+x^2} = \tan^{-1}x; \text{ and we get}$$

$$(2)\qquad \int_{-1}^{1} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1),$$

which is the expression sought, for the area under P_0BP_n.

Now, we know from Trigonometry that

$$\tan^{-1}(1) = \frac{\pi}{4}, \text{ or } \frac{5\pi}{4}, \text{ or } \frac{9\pi}{4}, \ldots, \text{ or } (4k+1)\frac{\pi}{4};$$

and that

$$\tan^{-1}(-1) = -\frac{\pi}{4}, \text{ or } \frac{3\pi}{4}, \text{ or } \frac{7\pi}{4}, \ldots, \text{ or } (4k-1)\frac{\pi}{4};$$

where k may be any integer whatever.

The question: *what values, among the many which* $\tan^{-1}(1)$ *and* $\tan^{-1}(-1)$ *may have, should be chosen to give the correct value for the determinate area under* P_0P_n *in fig. 23?* — must admit a single, definite answer.

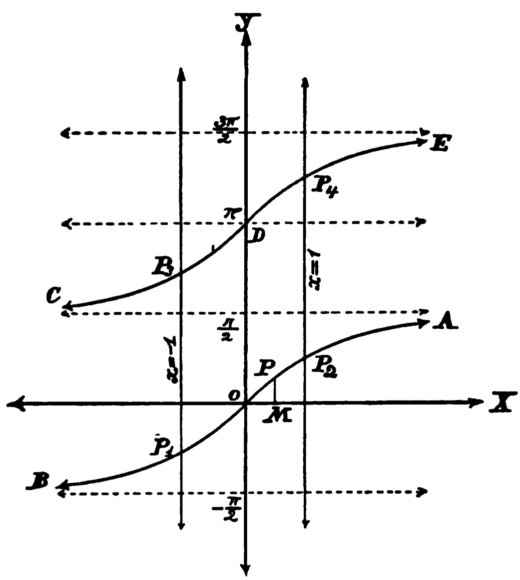

Fig. 24.

The principle which will supply this answer is furnished by considering the locus of the equation $y = \tan^{-1}x$. This locus is an infinite series of parallel curves, crossing the y-axis at $y_k = k\pi$, where k is any positive or negative integer. Each curve of the series is included between two lines parallel to the x-axis, and at a distance $= \pi$ apart (see fig. 24). Two of these curves are shown, —

BOA and CDE. Each curve extends without limit to the right and left, approaching the dotted lines as asymptotes.

Observe, now, that the x, in equations (1) and (2), is required to increase *continuously* from -1 to 1; hence, on the locus of $y = \int \frac{dx}{1+x^2} = \tan^{-1}x$, the ordinate, $y = \tan^{-1}x = MP$, of a point P, must change value *continuously* from $y_0 = \tan^{-1}(-1)$ to $y_n = \tan^{-1}(1)$.

This it cannot do unless the point P remains on a single curve, as on BOA, while the abscissa x increases from -1 to 1. This may be seen by observing that, on curve BOA, point P is at P_1 when $x = -1$ and at P_2 when $x = 1$; that, on curve CDE, point P is at P_3 when $x = -1$ and at P_4 when $x = 1$; and that, *it is impossible for point P (whose ordinate is $y = tan^{-1}x$) to pass from curve BOA to curve CDE when x varies continuously from -1 to 1*; that is, it is impossible for $y = \tan^{-1}x$ to change, *continuously*, from its value at P_1 to its value at P_4.

Hence we get the principle: —

*The two values, the one of $tan^{-1}(-1)$, the other of $tan^{-1}(1)$, must be taken on the same *branch of the series of curves representing $y = tan^{-1}x$.*

At P_1 (fig. 24) $x_0 = -1$, and $y_0 = \tan^{-1}(-1) = -\frac{\pi}{4}$. At P_2, $x_n = 1$, and $y_n = \tan^{-1}(1) = \frac{\pi}{4}$. Hence, we get, for the area sought (fig. 23)

$$(3) \qquad \int_{-1}^{1} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

The same value for the area is obtained if we use any other branch of the series. For example, at P_3, $x_0 = -1$, $y_0 = \frac{3\pi}{4}$; and at P_4, $x_n = 1$, $y_n = \frac{5\pi}{4}$. Hence, using these values, we get

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-1}^{1} = \frac{5\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{2}.$$

* Any single curve, of the infinite series of curves, may be called a *branch*.

The same considerations apply, in the case of the anti-differential $\int \frac{dx}{1+x^2}$, whatever be the limits of integration. In particular,

$$(4)\qquad \int\limits_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi, \text{ (see § 39)},$$

which gives the entire area under the curve in fig. 23.

It may be observed that the single branch BOA is sufficient for all cases; that is, *for any integral,* $\int\limits_a^b \frac{dx}{1+x^2}$, *it will be sufficient to take the values of* $\tan^{-1}a$ *and* $\tan^{-1}b$ *on the branch* BOA*; since* x *traverses its entire range of values from* $-\infty$ *to* $+\infty$ *on* BOA*, which lies between* $y_1 = -\frac{\pi}{2}$ *and* $y_2 = \frac{\pi}{2}$.

Problem II. $\displaystyle\int\limits_a^b \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}\Big|_a^b.$

This problem may be treated precisely as the preceding. In this case the anti-differential is $y = \frac{1}{c}\tan^{-1}\frac{x}{c}$, and its locus consists of an infinite series of parallel curves of the same character and relative positions as those in fig. 24; the differences being, that the curves cross the y-axis at $y_k = \frac{k\pi}{c}$ (where k is any integer), and that the branches lie in strips of the plane included between lines parallel to the x-axis, whose distance apart is $\frac{\pi}{c}$ (compare ll. 11–15, p. 85).

Problem III. $\displaystyle\int\limits_a^b \frac{dx}{\sqrt{c^2-x^2}} = \sin^{-1}\frac{x}{c}\Big|_a^b.$

In this case the locus of the anti-differential is the sinusoid, or harmonic curve, following the y-axis in both directions without end. (See fig. 25.*) The limits, a and b, must, each, be not greater

* Fig. 25 needs to be corrected as follows:

(1) the P_3 nearest to point D should be P_3';

(2) the equations of the lines, BE and CG, should be $\frac{x}{c} = -1$, and $\frac{x}{c} = 1$, respectively.

than c; and either one, or both, may be negative. The locus of $y = \sin^{-1}\frac{x}{c}$, like that of $y = \tan^{-1}x$, consists of infinitely many branches: but, unlike that of $y = \tan^{-1}x$, its branches are joined together so as to form one continuous locus. One branch is $ABOCD$, from $y = -\pi$ to $y = \pi$; another is $DEFGH$, from $y = \pi$ to $y = 3\pi$. The line $x = a$, cuts each branch in two points, as P'_1 and P_1: the line $x = b$, also, cuts each branch in two points, as P_2 and P'_2.

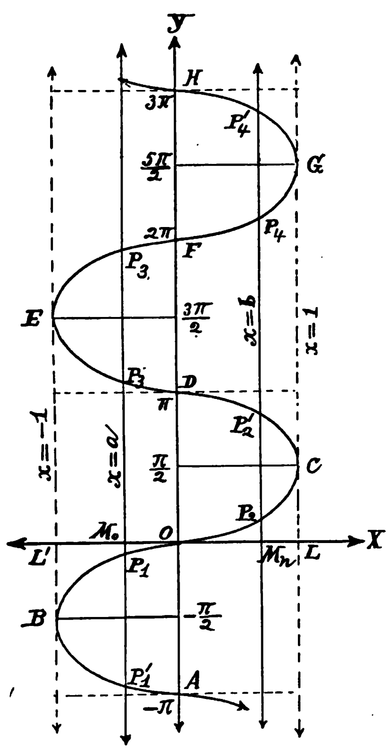

Fig. 25.

The continuous variation of x, from a to b, requires $y = sin^{-1}\frac{x}{c}$ to change continuously, on one branch; and, further, that y should

vary between two adjacent points, as P_1 *and* P_2. Hence, in determining the given integral

$$(5) \quad \int_a^b \frac{dx}{\sqrt{c^2 - x^2}} = \sin^{-1}\frac{x}{c}\Bigg|_a^b = \sin^{-1}\frac{b}{c} - \sin^{-1}\frac{a}{c},$$

we should have to take either, $y_0 = \sin^{-1}\frac{a}{c}$ at P_1, and $y_n = \sin^{-1}\frac{b}{c}$ at P_2; or, $y_0 = \sin^{-1}\frac{a}{c}$ at P_3, and $y_n = \sin^{-1}\frac{b}{c}$ at P_4.

As in problems I and II, *all branches but the simplest*, viz.: *ABOCD, may be ignored;* and the values of the anti-differential can be taken as the two *adjacent* ones on *ABOCD*. Indeed, it may be seen that *only the portion BOC, of the one branch, is needed;* since, on it, the variable $\frac{x}{c}$ traverses its entire range of possible variation from -1 to 1. Hence, *in evaluating the integral* $\int_a^b \frac{dx}{\sqrt{c^2 - x^2}} = \sin^{-1}\frac{x}{c}\Big|_a^b$, *it will be sufficient to take for* $\sin^{-1}\frac{b}{c}$ *and* $\sin^{-1}\frac{a}{c}$ *the values which they have between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.

In particular, if $a = -\frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$, and $c = 1$, we get

$$\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}x\Bigg|_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}.$$

41. VOLUMES OF SOLIDS OF ROTATION.

Let the equation of the curve AB, in fig. 26, be $y = f(x)$. Let P_0 and P_n be two fixed points on it, whose ordinates are M_0P_0 and M_nP_n. Suppose the area $M_0M_nP_nP_0$ to be rotated once around OX: a solid of rotation will be generated, whose lateral surface is the surface generated by the arc P_0P_n; and whose bases are the circles generated by the ordinates M_0P_0 and M_nP_n. *It is required to define, and calculate, the* **Volume** *of this solid.*

As in § 31, let the series of equi-basal rectangles, as shown in fig. 26, be inscribed in the area in question. The base of each is $\Delta x = dx = \frac{x_n - x_0}{n}$, and their altitudes are the ordinates of the points $P_0, P_1, P_2, \ldots . P_{n-1}$; that is, $f(x_0), f(x_1), f(x_2), \ldots . f(x_{n-1})$.

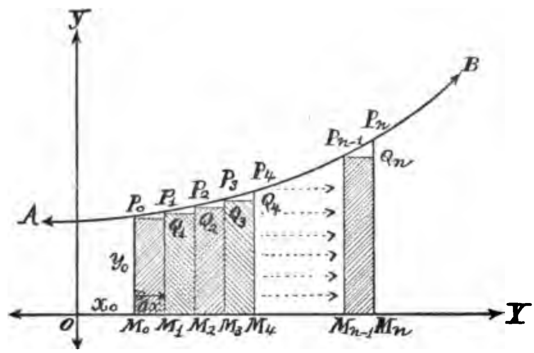

Fig. 26.

The sum of their areas is

$$(1) \qquad U = \sum_{x_0}^{x_{n-1}} f(x)\, dx, \qquad [\text{Eq. (6), § 31}];$$

and the limit of the sum of these rectangles has been defined as the area under P_0P_n.

Any of these rectangles, as M_0Q_1, will generate a cylinder of rotation, whose altitude is $M_0M_1 = dx$, whose radius is $M_0P_0 = f(x_0)$, and whose volume [see Elementary Geometry] is $\pi[f(x_0)]^2dx$. Hence, we may easily obtain the expression,

$$(2) \qquad \pi[f(x_0)]^2dx + \pi[f(x_1)]^2dx + \ldots . + \pi[f(x_{n-1})]^2dx$$
$$= \sum_{x_0}^{x_{n-1}} \pi[f(x)]^2dx,$$

for the sum of the volumes generated by the n rectangles inscribed under P_0P_n.

Since the limit of the sum of the areas of the rectangles, when n is indefinitely increased, is the area under the arc P_0P_n; *we shall define the* **Volume** *generated by the area under* P_0P_n *as the limit of*

the sum of the volumes generated by the n inscribed rectangles, when n is indefinitely increased.

Call this volume F; then, from (2), we have to find the value of the expression,

$$(3) \qquad F = \lim_{n=\infty} \sum_{x_0}^{x_n} \pi[f(x)]^2 dx.$$

Now, this problem is identical with the following: *Find the area under the curve $y = \pi[f(x)]^2$, from x_0 to x_n.* For, from equation (7) § 31, this area is

$$(4) \qquad G = \lim_{n=\infty} \sum_{x_0}^{x_n} \pi[f(x)]^2 dx.$$

That is, the *volume* generated by rotating, about the x-axis, the arc of $y = f(x)$ from x_0 to x_n, is numerically equal to the *area* under $y = \pi[f(x)]^2$, from x_0 to x_n.

It follows, therefore, from equation (16) § 33, that

$$(5) \qquad F = \int_{x_0}^{x_n} \pi[f(x)]^2 dx = \pi \int_{x_0}^{x_n} y^2 dx,$$

which is a perfectly general expression for the volume generated by rotating, about the x-axis, the arc of the curve, $y = f(x)$, from x_0 to x_n.

In like manner, it may be proved that

$$(6) \qquad F = \int_{y_0}^{y_n} \pi[f(y)]^2 dy = \pi \int_{y_0}^{y_n} x^2 dy,$$

is the general expression for the volume generated by rotating, about the y-axis, the arc of the curve, $x = f(y)$, from y_0 to y_n.

The *differential element* of volume is

$$(7) \qquad dF = \pi y^2 dx,$$

when the rotation is about the x-axis; and is

$$(8) \qquad dF = \pi x^2 dy,$$

when the rotation is about the y-axis. In each case, it is a right circular cylinder, with the radius y or x, and with the infinitesimal altitude dx or dy.

Exercises.

1. Find the volume of the sphere generated by rotating the circle, $x^2 + y^2 = 2ax$, about the x-axis.

2. Find the volume of the prolate spheroid generated by rotating the ellipse, $b^2x^2 + a^2y^2 = a^2b^2$, about the x-axis: also, of the oblate spheroid generated by rotating this ellipse about the y-axis.

3. Find the volume of the paraboloid generated by rotating $y^2 = 4px$ about the x-axis, the altitude being $x_n = h$.

Compare this volume with that of the cylinder having the same base and altitude.

4. Find the volume generated by rotating the hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, about either axis. *Ans.* $\dfrac{32\pi a^3}{105}$.

5. Find the volume of the cycloidal spindle generated by rotating an arch of the cycloid,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta),$$

about its base. *Ans.* $5\pi^2 a^3$.

6. Show that the volume generated by an area lying between two curves, $y = f(x)$ and $y = \phi(x)$, when rotated about the x-axis, is

$$F = \pi \int_{x_0}^{x_n} \Big[\{f(x)\}^2 - \{\phi(x)\}^2 \Big] dx.$$

7. Find the volume of the elliptical anchor ring generated by rotating the ellipse, $y = c \pm \frac{b}{a}\sqrt{a^2 - x^2}$, about the x-axis. *Ans.* $2\pi^2 abc$.

If $a = b$, this becomes a circular anchor ring, or torus, and $F = 2\pi^2 a^2 c$.

If, also, $c = b = a$, the ring has no opening, a cross section through its axis will be two equal tangent circles, and $F = 2\pi^2 a^3$.

8. Find the volume generated by rotating $x^2y^2 = (a - x)(x - b)$ about the x-axis. *Ans.* $2\pi(b - a) + \pi(b + a)\log\dfrac{a}{b}$.

9. Find the volume generated by rotating the catenary, $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, about the x-axis, taking the arc between the limits $-x_1$ and x_1.

Ans. $\dfrac{\pi a^2}{2}\left[\dfrac{a}{2}\left(e^{\frac{2x_1}{a}} - e^{-\frac{2x_1}{a}}\right) + 2x_1\right]$.

CHAPTER XVI.

DEFINITIONS AND PRINCIPLES: ROLLE'S THEOREM: LAW OF THE MEAN, AND DEPENDENT THEOREMS: GENERALIZED LAW OF THE MEAN.

42. DEFINITIONS AND PRINCIPLES.

I. We shall use the phrase, **the interval** (a, b), to refer to the set of values from a to b, inclusive; where a and b are arbitrary constants; and, unless the contrary is stated, $a < b$.

The value x_1, will be said to **belong** to the interval (a, b) when $a \leqq x_1 \leqq b$: and the interval (c, d) will be **contained in** the interval (a, b) when c and d belong to the interval (a, b).

II. The function $f(x)$, is **continuous at the value** $x = x_1$ if it is single-valued and

$$(1)\quad \lim_{\Delta x \doteq 0}\Big[f(x_1 - \Delta x) - f(x_1)\Big] = \lim_{\Delta x \doteq 0}\Big[f(x_1 + \Delta x) - f(x_1)\Big] = 0;$$

that is, if

$$(2)\qquad \lim_{\Delta x \doteq 0} f(x_1 - \Delta x) = \lim_{\Delta x \doteq 0} f(x_1 + \Delta x) = f(x_1).$$

If $f(x_1) = \infty$, it is possible for $f(x)$ to be continuous *up to* x_1, either on one side, or on both sides, of x_1; but $f(x)$ is not continuous *at* x_1. For example, $y = \dfrac{x}{(x - x_1)^2}$, and $y = \dfrac{x}{x - x_1}$, are continuous *up to*, but not *at*, x_1.

III. The function $f(x)$, is continuous **within** the interval (a, b) if conditions (1) and (2) are satisfied when $a < x < b$; and it is continuous **throughout** the interval (a, b) if it is continuous within the interval and, also,

$$\lim_{\Delta x \doteq 0}\Big[f(a + \Delta x) - f(a)\Big] = 0, \text{ and } \lim_{\Delta x \doteq 0}\Big[f(b - \Delta x) - f(b)\Big] = 0.$$

IV. The function $f(x)$, is said **to have a derivative on the left of x_1** if the ratio

$$\frac{f(x_1 - \Delta x) - f(x_1)}{-\Delta x}, \text{ when } \Delta x \doteq 0,$$

either α) approaches a limit, or β) becomes determinately infinite; that is, becomes positively or negatively infinite: and $f(x)$ is said to have a derivative **on the right of** x_1 if the ratio

$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}, \text{ when } \Delta x \doteq 0,$$

either approaches a limit, or becomes determinately infinite. These may be called the *left-hand*, and *right-hand*, derivatives.

The function $f(x)$, is said to have a derivative **at x_1**, when it has a derivative *on both sides of x_1;* that is, when

$$(3)\quad \lim_{\Delta x \doteq 0}\left[\frac{f(x_1 - \Delta x) - f(x_1)}{-\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\right] = D_x f(x)\Big|_{x = x_1}.$$

Condition (3) *excludes the case of the cusp*, shown in fig. 10, p. 34; since the left-hand and right-hand derivatives at P_1 and P_2 have opposite signs: but condition (3) *admits the case of the point of inflexion when the inflexion-tangent is parallel to the y-axis* (see § 21); since the derivatives on both sides of the point of inflexion become infinite with the same sign.

V. The function $f(x)$, is said to have a derivative **within** the interval (a, b) when it has a derivative for all values of x within the interval, but not including a and b; and to have a derivative **throughout** the interval (a, b) when it has a derivative within the interval, a right-hand derivative at $x = a$, and a left-hand derivative at $x = b$.

VI. If the function, $f(x)$, is single-valued and has a derivative at the value $x = x_1$, it is continuous at x_1; for, by hypothesis,

$$(4)\quad \lim_{\Delta x \doteq 0}\left[\frac{f(x_1 - \Delta x) - f(x_1)}{-\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}\right];$$

whence,

$$\lim_{\Delta x \doteq 0}\Big[f(x_1 - \Delta x) - f(x_1)\Big] = \lim_{\Delta x \doteq 0}\Big[f(x_1 + \Delta x) - f(x_1)\Big] = 0;$$

and condition (1), of this §, is satisfied at x_1.

The converse of this proposition, viz.: *If $f(x)$ is continuous at x_1 it has a derivative at x_1*, is true for all the functions the student is likely to find in his work; but *is not universally true.*

VII. If the function, $f(x)$, is continuous and single-valued throughout the interval (a, b); and if, moreover, $f(a)$ and $f(b)$ have opposite signs; then there must be at least one value of x, between a and b, at which $f(x)$ vanishes; that is,

$$\text{if } f(a) \gtrless 0 \text{ and } f(b) \lessgtr 0, \text{ then } \quad f(x_1) = 0, \quad a < x_1 < b.$$

The analytic proof of this proposition is too long for insertion here: it can readily be seen to be true by plotting the locus of $y = f(x)$ for the interval (a, b).

VIII. We shall find it convenient to introduce the Lagrange symbols for the successive derivatives of $f(x)$; viz. $f'(x) \equiv D_x f(x)$, $f''(x) \equiv D_x^2 f(x)$, $f'''(x) \equiv D_x^3 f(x)$, ... $f^n(x) \equiv D_x^n f(x)$.

With this notation, the symbol $f'(x_1)$ represents the particular value which $f'(x)$ assumes when $x = x_1$; that is, the two symbols, $f'(x_1)$ and $D_x f(x)\Big|_{x=x_1}$, mean precisely the same thing. The convenience of the first is obvious.

43. ROLLE'S THEOREM.

Suppose the function, $f(x)$, to have a derivative and to be finite throughout the interval (a, b); suppose also that $f(x)$ vanishes at a and b, that is,

(1) $$f(a) = f(b) = 0;$$

then, there must be at least one value of x, between a and b, for which $f'(x)$ vanishes; that is,

(2) $$f'(x_1) = 0, \qquad a < x_1 < b.$$

This is known as Rolle's Theorem. It means, geometrically, that: *If a single-branched curve containing no cusp, connects two points, A and B, on the x-axis, the tangent to the curve must be parallel to the x-axis for at least one point, P_1, between A and B.*

Proof:—The function $f(x)$ must be continuous, by § 42, VI. Then, either (1) $f(x) = 0$ throughout the interval (a, b), or (2) $f(x)$ takes only positive values, or (3) $f(x)$ takes only negative values, or (4) $f(x)$ takes both positive and negative values.

In case (1), $f(x)$ is a constant; and, therefore, its derivative, $f'(x)$, vanishes by Ex. 1*b*, § 9, for all values of x in the interval; and the theorem holds.

In case (2), $f(x)$ must begin by increasing from $f(a)=0$; and must end by decreasing to $f(b)=0$; as x increases from a to b. *There must, then, be at least one value of x, say x_1, for which $f(x)$ is a maximum;* that is, at which

$$(3) \qquad f(x_1-\Delta x) < f(x_1) > f(x_1+\Delta x).$$

Hence, both

$$f(x_1-\Delta x)-f(x_1) \text{ and } f(x_1+\Delta x)-f(x_1)$$

are negative; and, therefore, the ratios,

$$(4) \qquad \frac{f(x_1-\Delta x)-f(x_1)}{-\Delta x} \text{ and } \frac{f(x_1+\Delta x)+f(x_1)}{\Delta x},$$

must have opposite signs.

But $f(x)$ has a derivative at x_1, by hypothesis; hence, by § 42, IV, (3), the two ratios in (4) must have identical limits as $\Delta x \doteq 0$, and this limit is $f'(x_1)$. *Now, it is obvious that two variables having opposite signs can have no common limit except zero;* hence,

$$(5) \qquad f'(x_1)=0,$$

and the theorem holds for case (2).

Similarly, case (3) can be proved; since $f(x)$ must take at least one minimum value when $f(x)$ takes only negative values.

Case (4) follows at once from cases (2) and (3); for, suppose $f(x)$ first takes positive values then negative values. Since $f(x)$ is continuous, it must vanish when it changes sign (§ 42, VII). Let $x=b_1$ be the value of x at which $f(b_1)=0$; then $a<b_1<b$, and by case (2) $f'(x)$ must vanish for at least one value of x between a and b_1; which proves the theorem for case (4).

Exercise.

1. Write out the proof for case (3).

44. THE LAW OF THE MEAN.

If the function, $f(x)$, has a derivative throughout the interval (a, b), then there must be at least one value, x_1, between a and b, at which we have

$$(1) \qquad \frac{f(b)-f(a)}{b-a}=f'(x_1) \quad , \quad a<x_1<b.$$

This theorem, called *Law of the mean*, may be proved by aid of the following auxiliary function,

$$(2) \quad F(x) \equiv f(x)-f(a)-(x-a)\frac{f(b)-f(a)}{b-a};$$

which, obviously, has a derivative throughout the interval (a, b). Now, $F(a) = F(b) = 0$; for,

$$F(a)=f(a)-f(a)-(a-a)\frac{f(b)-f(a)}{b-a}=0;$$

and

$$F(b)=f(b)-f(a)-(b-a)\frac{f(b)-f(a)}{b-a}=0.$$

Hence, by Rolle's Theorem, $F'(x)$ must vanish for some value, x_1, between a and b.

$$(3) \quad \therefore\ F'(x_1)=f'(x_1)-\frac{f(b)-f(a)}{b-a}=0, \ a<x_1<b.$$

Equation (3) is obtained by differentiating (2), then substituting x_1 for x.

Transposing in (3) we get (1), *q. e. d.*

Another convenient form of (1) may be obtained as follows: —

Let the interval $(x, x+h)$ be contained in the interval (a, b). The theorem, above, must hold for this new interval; since all the initial conditions are satisfied in it. We get, therefore, from (1),

$$(4) \qquad \frac{f(x+h)-f(x)}{h}=f'(x+\theta h), \ 0<\theta<1;$$

since $\qquad x<x+\theta h<x+h$, when $0<\theta<1$.

Clearing (4) and transposing we get,

$$(5) \qquad f(x+h)=f(x)+hf'(x+\theta h), \ 0<\theta<1;$$

which is a common form of the "Law of the mean."

Exercises.

1. Show that $\dfrac{f(b)-f(a)}{b-a}$ is the slope of the chord joining the two points, A and B, whose abscissæ are $x=a$, $x=b$, on the locus of $y=f(x)$.

2. Verify equation (1) for the function $y=4\sqrt{x}$ for the interval $(4, 9)$; and show that (1) means that: *at some point on the parabola $y=4\sqrt{x}$, between the points $x=4$ and $x=9$, the tangent is parallel to the chord joining these points.* Can you find the point of contact of that tangent? Find the value of θ.

3. Show that the geometric meaning of the "Law of the mean" is this:—
If A and B are two points on the locus of $y=f(x)$, having the abscissæ a and b, then, provided this locus has a tangent at each point and has no cusp, there is one point, at least, between A and B at which the tangent is parallel to the chord AB.

4. Show that the function, $F(x)$, in equation (2), represents the length PQ, where P and Q are points, respectively, on the curve $y=f(x)$ and on the chord AB, [see ex. 3] having the same abscissa x, $a<x<b$.
Show that the length, $F(x)$, is a maximum, or a minimum, at a point where the tangent is parallel to the chord.

45. THEOREMS DEDUCED FROM THE LAW OF THE MEAN.

I. *If the derivative, $f'(x)$, of $f(x)$ equals a constant, k, throughout the interval (a, b), then $f(x)$ is a linear function of x in this interval.*

Proof;—Since $f'(x)=k$, $f(x)$ has a derivative; and we get, by the Law of the mean,

$$(1) \qquad \frac{f(x)-f(a)}{x-a}=f'(x_1), \quad a<x_1<x \leqq b,$$
$$= k, \text{ by hypothesis.}$$

$$(2) \qquad \therefore f(x)=f(a)+k(x-a), \quad a \leqq x \leqq b.$$

The right member of (2) is linear, obviously; hence, since (2) is an identity, $f(x)$ is linear, *q. e. d.*

II. *If the derivative, $f'(x)$, vanishes throughout the interval (a, b), then $f(x)=f(a)=A$ is a constant in this interval.*

Proof:—This is the special case of I, above, in which $k=0$. Substituting in (2) we get, since $x-a$ is finite,

$$(3) \qquad f(x)=f(a)=A.$$

III. *If the derivatives,* $f'(x)$ *and* $\phi'(x)$, *of two functions,* $f(x)$ *and* $\phi(x)$, *are equal for the same values of* x *throughout the interval* (a, b); *then* $f(x)$ *and* $\phi(x)$ *can differ only by a constant term.*

Proof: — $f'(x) = \phi'(x)$, by hypothesis;

$$\therefore f'(x) - \phi'(x) = 0, \quad a \leqq x \leqq b.$$

But $f'(x) - \phi'(x) = D_x [f(x) - \phi(x)] = 0$;
hence, by II, above, we must have,

$$(4) \qquad f(x) - \phi(x) = A; \text{ or}$$

$$(5) \qquad f(x) = \phi(x) + A, \quad a \leqq x \leqq b,$$

where A is any constant.

[This theorem is assumed in § 24, equations (15), (16).]

46. GENERALIZED LAW OF THE MEAN.

I. *Let* $f(x)$ *and* $\phi(x)$ *be two functions which are continuous throughout the interval* (a, b), *and have finite derivatives everywhere within this interval; suppose, also, that* $\phi'(x)$ *does not vanish, and* $\phi(b)$ *is not equal to* $\phi(a)$; *then, for some value,* x_1, *between* a *and* b, *we can prove that,*

$$(1) \qquad \frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(x_1)}{\phi'(x_1)}, \quad a < x_1 < b.$$

Proof: — Denote the constant, $\dfrac{f(b) - f(a)}{\phi(b) - \phi(a)}$, by G. Then theorem (1) may be expressed,

$$(2) \qquad G\phi'(x_1) - f'(x_1) = 0, \quad a < x_1 < b.$$

Consider the expression

$$(3) \qquad G\phi'(x) - f'(x), \quad a < x < b.$$

It is, obviously, the derivative of the function,

$$(4) \qquad F(x) \equiv G\phi(x) - f(x) + k, \quad a < x < b;$$

and, from our hypothesis, (3) must have a determinate value for every value of x between a and b: so that $F(x)$ has a derivative, $F'(x) \equiv G\phi'(x) - f'(x)$, for all values of x between a and b. The constant, k, is arbitrary. Let it be chosen so that $F(x)$ shall vanish when $x = a$; then,

$$(5) \qquad F(a) = G\phi(a) - f(a) + k = 0;$$

whence, $\qquad k = f(a) - G\phi(a).$

Putting this value of k in (4), factoring, and restoring the value of G, we get,

$$(6)\quad F(x) \equiv \frac{f(b)-f(a)}{\phi(b)-\phi(a)}\,[\phi(x)-\phi(a)]-f(x)+f(a).$$

From (6) it is plain that $F(x)$ *vanishes when* $x=a$ *and when* $x=b$; and, therefore, by Rolle's Theorem, we must have,

$$(7)\qquad F'(x_1) \equiv G\,\phi'(x_1) - f'(x_1) = 0,\ a<x_1<b.$$

Whence we get,

$$(8)\qquad \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = G = \frac{f'(x_1)}{\phi'(x_1)},\ a<x_1<b,\ q.\ e.\ d.$$

If the interval $(x, x+h)$ is contained in the interval (a, b), and $\phi(x) \gtrless \phi(x+h)$; then (8) may be put into the form,

$$(9)\qquad \frac{f(x+h)-f(x)}{\phi(x+h)-\phi(x)} = \frac{f'(x+\theta h)}{\phi'(x+\theta h)},\ 0<\theta<1;$$

since $\qquad x<x+\theta h<x+h$, when $0<\theta<1$.

The following extension of (9) is sometimes useful:

Suppose that $f'(x)$ *and* $\phi'(x)$ *are continuous throughout the interval* $(x, x+h)$, *which is contained in the interval* (a, b); *suppose that* $\phi'(x) \gtrless \phi'(x+h)$, *and that* $\phi''(x)$ *does not vanish, and that, moreover,* $\phi''(x)$ *and* $f''(x)$ *are finite for all values of* x *between* x *and* $x+h$; *then, as above, we may prove that,*

$$(10)\quad \frac{f'(x+h)-f'(x)}{\phi'(x+h)-\phi'(x)} = \frac{f''(x+\theta h)}{\phi''(x+\theta h)},\ 0<\theta<1.$$

And, in general, if the $(n-1)^{th}$ derivatives, $f^{n-1}(x)$ and $\phi^{n-1}(x)$, with their derivatives, $f^n(x)$ and $\phi^n(x)$, satisfy conditions identical with those of the generalized Law of the Mean; then

$$(11)\quad \frac{f^{n-1}(x+h)-f^{n-1}(x)}{\phi^{n-1}(x+h)-\phi^{n-1}(x)} = \frac{f^n(x+\theta h)}{\phi^n(x+\theta h)},\ 0<\theta<1.$$

Exercises.

1. Verify equation (8) for the case in which $f(x) = x^2 + 2x$, $\phi(x) = x^3 + x + 1$, $a = 2$, and $b = 3$. Find x_1. Use form (9) also, and find θ.

2. Find x_1 in eq. (8) when $f(x) = \cos x$, $\phi(x) = \sin x$, $a = 0$, and $b = \frac{\pi}{6}$.

CHAPTER XVII.

THE INDETERMINATE FORMS,

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \times \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

47. LIMITING VALUES OF CERTAIN FUNCTIONS WHICH ASSUME INDETERMINATE FORMS.

A. The form $\frac{0}{0}$.

Let $F(x) = \frac{f(x)}{\phi(x)}$ be a function of x which, for $x = a$, takes the indeterminate form,

$$(1) \qquad F(a) = \frac{f(a)}{\phi(a)} = \frac{0}{0}, \qquad f(a) = \phi(a) = 0.$$

In many cases the limiting value may be found by canceling a factor of $f(x)$ and $\phi(x)$, whose presence causes each to vanish; and then substituting in the reduced form the value of x which made $f(x)$ and $\phi(x)$ vanish. For example,

$$F(x) = \frac{\cos x}{\operatorname{ctn} x} \quad \text{is indeterminate if } x = \frac{\pi}{2}.$$

But,
$$F(x) = \frac{\cos x}{\operatorname{ctn} x} = \frac{\cos x}{\frac{\cos x}{\sin x}} = \sin x;$$

$$\therefore F\left(\frac{\pi}{2}\right) = \frac{\cos \frac{\pi}{2}}{\operatorname{ctn} \frac{\pi}{2}} = \sin \frac{\pi}{2} = 1.$$

The following theorems will, generally, furnish the limiting value of $F(x) = f(x) \div \phi(x)$, when $f(x)$ and $\phi(x)$ are simultaneously zero.

I. *Let $f(x)$ and $\phi(x)$ satisfy the conditions of theorem* I *of* § 46 *for the interval* $(a, a+h)$; *and let* $f(a) = \phi(a) = 0$; *then, if the ratio* $\dfrac{f'(x)}{\phi'(x)}$ *approaches a limit,* A, *as* x *approaches* a, *we shall have*

$$(2) \qquad F(a) = \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} = A.$$

In this case, equation (1) of § 46 becomes, on replacing b by $a+h$, and noticing that $f(a) = \phi(a) = 0$,

$$(3) \qquad \frac{f(a+h)}{\phi(a+h)} = \frac{f'(x_1)}{\phi'(x_1)}, \quad a < x_1 < a+h.$$

Now, if $h \doteq 0$, $x_1 \doteq a$; and (3) approaches the limiting form,

$$(4) \qquad \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

But, by hypothesis, the right member of (3) approaches the limit, A, when x_1 approaches a; hence, (4) shows that

$$(5) \qquad F(a) = \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} = A, \qquad q.\ e.\ d.$$

It is assumed that a *is finite.*

Exercises.

Find the limiting values of the following indeterminate forms for the values of x given with each: —

1. $\dfrac{\sin mx}{x}$, when $x = 0$;
2. $\dfrac{e^x - 1}{x}$, when $x = 0$;
3. $\dfrac{1 - \cos x}{\sin^2 x}$, when $x = 0$;
4. $\dfrac{\log x}{x^2 - 1}$, when $x = 1$;
5. $\dfrac{\sin x}{\sin 2x}$, when $x = 0$;
6. $\dfrac{a^x - 1}{x a^x}$, when $x = 0$.

II. *In some cases when* $f(a) = \phi(a) = 0$ *in* $F(x) = \dfrac{f(x)}{\phi(x)}$, *it happens that both* $f'(a) = 0$ *and* $\phi'(a) = 0$; *so that* $\dfrac{f'(a)}{\phi'(a)}$ *is also indeterminate.*

In this case, we get, from (10) § 46,

$$(6) \qquad \frac{f'(a+h)}{\phi'(a+h)} = \frac{f''(x_2)}{\phi''(x_2)}, \quad a < x_2 < a+h;$$

provided $f'(x)$ and $\phi'(x)$, with their derivatives, $f''(x)$ and $\phi''(x)$, satisfy the same conditions as prescribed in § 46 for $f(x)$ and $\phi(x)$, etc.

In (6), if we make $h \doteq 0$, $x_2 \doteq a$, and we get the limiting form,

$$(7) \qquad \frac{f'(a)}{\phi'(a)} = \frac{f''(a)}{\phi''(a)}.$$

Therefore, if the ratio $\dfrac{f''(x)}{\phi''(x)}$ approaches the limit A, the ratio $\dfrac{f'(x)}{\phi'(x)}$ must, from (7), approach the same limit when $x \doteq a$; and, *since* $\dfrac{f'(x)}{\phi'(x)}$ *has thus been shown to approach a limit*, A, *it follows from theorem I that* $F(x) = \dfrac{f(x)}{\phi(x)}$ *approaches this limit.*

We get, therefore, when $f(a) = \phi(a) = 0$, and $f'(a) = \phi'(a) = 0$,

$$(8) \qquad F(a) = \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} = \frac{f''(a)}{\phi''(a)}.$$

If it should happen that both $f''(a) = 0$ and $\phi''(a) = 0$, *we could show in like manner that, when the ratio,* $\dfrac{f'''(a)}{\phi'''(a)}$, *has a determinate value, we must have*

$$(9) \qquad F(a) = \frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} = \frac{f''(a)}{\phi''(a)} = \frac{f'''(a)}{\phi'''(a)}.$$

This reasoning may be extended in case $f'''(a)$ and $\phi'''(a)$ should, also, equal zero.

It may be shown that (5), (8), and (9), hold, also, when $a = \infty$.

Exercises.

1. $\dfrac{e^x + e^{-x} - 2}{x \sin x}$, when $x = 0$.

2. $\dfrac{\tan ax - ax}{\tan bx - bx}$, when $x = 0$.

3. $\dfrac{x - \sin x}{x^3}$, when $x = 0$.

4. $\dfrac{\tan x - \sin x}{x - \sin x}$, when $x = 0$.

5. $\dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$, when $x=1$.

6. $\dfrac{x\sqrt{3x-2x^4}-x\sqrt[5]{x}}{1-\sqrt[3]{x^2}}$, when $x=1$.

7. $\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$, when $x=a$.

B. *The form* $\frac{\infty}{\infty}$.

Let $F(x)=\dfrac{f(x)}{\phi(x)}$ be a function of x which, for a value $x=a$, takes the form,

$$F(a)=\frac{f(a)}{\phi(a)}=\frac{\infty}{\infty},\ f(a)=\infty,\ \phi(a)=\infty,$$

where a may be finite or infinite.

In this case, as in the form $\dfrac{0}{0}$, *if the ratio,* $\dfrac{f'(a)}{\phi'(a)}$, *takes a determinate value,* A, *when* x *approaches* a, *then we may prove that,*

$$(10)\qquad F(a)=\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}=A;$$

or, if $\dfrac{f'(a)}{\phi'(a)}$ *is indeterminate but* $\dfrac{f''(a)}{\phi''(a)}$ *has a determinate value,* A,

$$(11)\qquad F(a)=\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}=\frac{f''(a)}{\phi''(a)}=A.$$

If $\dfrac{f''(a)}{\phi''(a)}$ is also indeterminate the higher derivatives may be used, as in the case of $\dfrac{0}{0}$.

The proof of the foregoing rule is omitted.

C. *The form* $0\times\infty$.

Let $F(x)=f(x)\times\phi(x)$ be a function of x which, for a value $x=a$ takes the indeterminate form,

$$(12)\quad F(a)=f(a)\times\phi(a)=0\times\infty,\ f(a)=0,\ \phi(a)=\infty.$$

In this case we may put $F(x)$ into the form,

(13) $$F(x) = \frac{f(x)}{\dfrac{1}{\phi(x)}}; \text{ or, } F(x) = \frac{\phi(x)}{\dfrac{1}{f(x)}}.$$

The first of these gives $F(a) = \frac{0}{0}$; and the second gives $F(a) = \frac{\infty}{\infty}$: hence, this form may generally be evaluated by the rule for the form $\frac{0}{0}$, or for $\frac{\infty}{\infty}$, after changing it to one of the forms in (13).

D. The form $\infty - \infty$.

Let $F(x) = f(x) - \phi(x)$ be a function of x which, for a value $x = a$, takes the indeterminate form,

(14) $$F(a) = f(a) - \phi(a) = \infty - \infty, f(a) = \infty, \phi(a) = \infty.$$

It frequently happens that $f(x) = f_1(x) \div f_2(x)$ and $\phi(x) = \phi_1(x) \div \phi_2(x)$; so that this form can be written,

(15) $$F(x) = f(x) - \phi(x) = \frac{f_1(x)}{f_2(x)} - \frac{\phi_1(x)}{\phi_2(x)}$$
$$= \frac{f_1(x)\,\phi_2(x) - f_2(x)\,\phi_1(x)}{\phi_2(x)\,f_2(x)},$$

in which $f_2(a) = \phi_2(a) = 0$. In such case, (15) shows that, in general, $F(x)$ takes the form $\frac{0}{0}$ after combining the fractions; hence, it can be treated as in case *A*.

In any case we may write,

(16) $$F(x) = f(x) - \phi(x) = \left[1 - \frac{\phi(x)}{f(x)}\right] f(x);$$

and, $F(a)$ must, therefore, be infinite if $\frac{\phi(a)}{f(a)}$ has any value different from unity.

If, however, $\frac{\phi(a)}{f(a)} = 1$, then (16) takes the form,

(17) $$F(a) = \left[1 - \frac{\phi(a)}{f(a)}\right] f(a) = 0 \times \infty;$$

and its limiting value can be found as in case *C*.

E. The forms 0^0, 1^∞, ∞^0.

Let $F(x) = [f(x)]^{\phi(x)}$. The following cases may arise when x takes the value, a:

α) $\quad F(x) = 0^0$, if $f(a) = \phi(a) = 0$;

β) $\quad F(x) = 1^\infty$, if $f(a) = 1$ and $\phi(a) = \infty$;

γ) $\quad F(x) = \infty^0$, if $f(a) = \infty$ and $\phi(a) = 0$.

Each of these cases may generally be evaluated by taking the logarithm of $F(x)$. We get

$$\log F(x) = \phi(x) \log f(x), \tag{18}$$

It may be seen from (18) that $\log F(a)$ takes the form $0 \times \infty$ for all three cases, α), β) and γ), above; hence, we may first find $\log F(a)$ as before; and thence obtain $F(a)$.

If $\log F(a) = k$, then

$$F(a) = [f(a)]^{\phi(a)} = e^k.$$

Exercises.

Find the limiting values of the following:

1. $\dfrac{\tan x}{\tan 3x}$, when $x = \dfrac{\pi}{2}$;

2. $x \log\left(1 + \dfrac{1}{x}\right)$, when $x = \infty$;

3. $(1 - \sin x) \tan x$, when $x = \dfrac{\pi}{2}$;

4. $\dfrac{1}{x-1} - \dfrac{1}{\log x}$, when $x = 1$;

5. $\sec x - \tan x$, when $x = \dfrac{\pi}{2}$;

6. $(1 + mx)^{\frac{1}{x}}$, when $x = 0$;

7. $\dfrac{x^3}{e^x}$, when $x = \infty$;

8. x^x, when $x = 0$.

9. $\left[1 + \dfrac{1}{x^2}\right]^x$, when $x = \infty$.

CHAPTER XVIII.

SERIES: CONVERGENCY: POWER SERIES.

48. SERIES: DEFINITIONS.

The term **series** is used to refer to a set of numbers such as

(1) $$u_1 + u_2 + u_3 + u_4 + u_5 + \dots \quad ,$$

whose **terms,** u_1, u_2, u_3, etc., are formed in some defined manner. The number of terms is unlimited, or infinite: and each term may be a constant, in which case it is called a **series of constant terms**; or, the terms may be all functions of some variable, in which case it is called a **series of variable terms.** A **series of positive terms** is one whose terms are all positive; and an **alternating series** is one whose terms are alternately positive and negative, either from the beginning or after a fixed term. A **power series** is one whose terms contain only positive integral powers of a variable, arranged in ascending order.

Each term of a series is assumed finite.

The symbol S_n, will be used for **the sum of the first** n **terms** of a given series: thus, $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, $\dots$,

(2) $$S_n = u_1 + u_2 + u_3 + u_4 + \dots + u_n.$$

It is understood that the terms must be added in the order in which they are written.

Each term of the series being finite, S_n *is finite for all finite values of* n*:* but, *if* n *is indefinitely increased*, it may happen that either (α) S_n approaches a limit, A; or, (β) S_n becomes infinite; or (γ) S_n is indeterminate.

In case (α) the series is said to be **convergent,** or **to converge to the limit,** A; but in cases (β) and (γ) the series is said to be **divergent.**

A series of constant terms must be either convergent or divergent; while a series of variable terms may be convergent for some values of their variable, but divergent for other values.

For example, the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$, is *convergent*;

since $$\lim_{n=\infty} S_n = \lim_{n=\infty}\left[\frac{3}{2} - \frac{1}{2\cdot 3^{n-1}}\right] = \frac{3}{2}:$$

but the series, $1 + 2 + 4 + 8 + \ldots$, is *divergent*;

since $\lim_{n=\infty} S_n = \lim_{n=\infty} [2^n - 1] = \infty$. The series, $3 - 3 + 3 - 3 + \ldots$, is *divergent*; since $\lim_{n=\infty} S_n$ is indeterminate.

The geometric series, $a + ax + ax^2 + ax^3 + \ldots$, is convergent if $-1 < x < 1$, divergent if $x = \pm 1$, and divergent if $-1 > x > 1$; since

$$\lim_{n=\infty} S_n = \lim_{n=\infty}\left[\frac{a(1-x^n)}{1-x}\right] = \frac{a}{1-x} - a \lim_{n=\infty}\left[\frac{x^n}{1-x}\right].$$

The arithmetic series, $a + (a + x) + (a + 2x) + \ldots$, is divergent for every value of x; since

$$\lim_{n=\infty} S_n = \lim_{n=\infty} [an + \tfrac{1}{2}n(n-1)x] = \infty,$$

unless both a and x equal zero.

When a formula can be found for S_n, as in the geometric and arithmetic series, it is generally a simple matter to determine whether the series is convergent or not: but in many important cases, no such formula can be obtained; so that other methods must be devised for determining the convergency or divergency of a given series.

In the case of a convergent series, the $\lim_{n=\infty} S_n = A$ can be substituted for the series in computing: *but a divergent series cannot be represented by any definite number, and must be avoided.*

A summary of the most useful rules for testing the divergency or convergency of series will be found in the next section. *The proofs of these rules are omitted.*

NOTE. — The student will find demonstrations of nearly all the theorems stated in §§ 49–50 in C. Smith's *Treatise on Algebra*, in G. Chrystal's *Text-book of Algebra*, and in W. F. Osgood's *Introduction to Infinite Series.*

49. CONDITIONS AND TESTS OF CONVERGENCY AND DIVERGENCY OF SERIES.

I. The series, $u_1 + u_2 + u_3 + u_4 + \dots$, is convergent if $\lim_{n=\infty} S_n = \lim_{n=\infty} [u_1 + u_2 + u_3 + \dots + u_n] = A$, where A is some determinate number; the series is divergent if S_n is infinite or indeterminate when n is infinite. [See § 48].

II. The series, $u_1 + u_2 + u_3 + u_4 + \dots$, cannot be convergent *unless* $\lim_{n=\infty} u_n = 0$; but it is *not always convergent even when* $\lim_{n=\infty} u_n = 0$.

III. The convergency or divergency of any series cannot be affected by omitting, or affixing, a finite number of terms at the beginning; or by reversing all its signs.

A. *Series whose terms are all of like sign.*

IV. Given the series to be tested,

$$(1) \qquad u_1 + u_2 + u_3 + u_4 + \dots ,$$

whose terms are all positive (or all negative). Compare it with a second series,

$$(2) \qquad v_1 + v_2 + v_3 + v_4 + \dots ,$$

whose terms are all positive (or all negative). *If series* (2) *is convergent and any term, v_i, is numerically greater than, or equal to, the corresponding term, u_i, of series* (1), *then series* (1) *is convergent.*

But, *if series* (2) *is divergent, and any term, v_i, is less than, or equal to, the corresponding term, u_i, of series* (1), *then series* (1) *is divergent.*

Two series which may be thus used for comparison are:

$$(3) \qquad a + ax + ax^2 + ax^3 + \dots ,$$

convergent if $-1 < x < 1$, *divergent* if $-1 \geqq x \geqq 1$;

and

$$(4) \qquad 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \dots ,$$

convergent if $x > 1$, *divergent* if $x \leqq 1$.

Exercises.

1. Show that the two series,

a) $$1 + \frac{1}{3^1} + \frac{1}{4^2} + \frac{1}{5^3} + \frac{1}{6^4} + \cdots\cdots,$$

b) $$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \cdots\cdots,$$

are *convergent.*

2. Show that the series,

$$\frac{1}{1+x} + \frac{1}{1+x^2} + \frac{1}{1+x^3} + \cdots\cdots,$$

is *convergent if* $x > 1$; *divergent if* $x \leqq 1$.

V. Given the series of all positive (or all negative) terms,

(5) $$u_1 + u_2 + u_3 + u_4 + \cdots\cdots$$

The ratio of the $(n+1)^{th}$ term to the n^{th} is $\frac{u_{n+1}}{u_n}$, where n may be $1, 2, 3, 4, \ldots\ldots \infty$. This ratio is called the **test-ratio** of the series.

Let $$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = \lambda.$$

If $0 \leqq \lambda < 1$, *the series is convergent*;
if $\lambda = 1$, *the series may be convergent or divergent*;
if $\lambda > 1$, *the series is divergent.*

In the case $\lambda = 1$ the following test will sometimes determine convergency or divergency:

VI. Denote the limit, $\lim_{n=\infty} \left[n\left(1 - \frac{u_{n+1}}{u_n}\right)\right]$, by k, where $u_{n+1} \div u_n$ has the same meaning as in V, *and all the terms of the series have the same sign.*

Then, if $k > 1$, or $k = \infty$, *the series is convergent;*
if $k = 1$, *the series may be convergent or divergent;*
if $k < 1$, or $k = -\infty$, *the series is divergent.*

The student should observe that this test will not apply unless the terms of the series all have the same sign.

B. *Series containing both positive and negative terms.*

VII. An alternating series,

$$(6) \qquad u_1 - u_2 + u_3 - u_4 + \ldots\ldots ,$$

will be convergent, if (α) $u_1 > u_2 > u_3 > u_4 > \ldots\ldots$,
and (β) $\lim\limits_{n=\infty} u_n = 0.$

VIII. A series containing both positive and negative terms, whether alternating or not, *will be convergent, if it is convergent after all the minus signs are changed to plus.*

IX. Any series,

$$(7) \qquad u_1 + u_2 + u_3 + u_4 + \ldots\ldots ,$$

will be

convergent if $-1 < \lambda < 1$,
divergent if $-1 > \lambda > 1$;

where

$$\lambda = \lim_{n=\infty}\left[\frac{u_{n+1}}{u_n}\right].$$

But if $\lambda = \pm 1$, this test will not determine convergency or divergency.

X. The infinite product,

$$(8) \qquad \left(1-\frac{k}{1}\right)\left(1-\frac{k}{2}\right)\left(1-\frac{k}{3}\right)\left(1-\frac{k}{4}\right)\ldots\ldots \text{ to infinity,}$$

converges to the limit zero for all values of k greater than 0.

Exercises.

1. Show that the series,

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots ,$$

is *convergent* if $-1 < x < 1$, and *divergent* if $-1 > x > 1$. Can you determine the cases $x = \pm 1$?

2. Show that the following series are *convergent for all finite values of x*:

$$a) \qquad 1 + x\log a + \frac{x^2\log^2 a}{2!} + \frac{x^3\log^3 a}{3!} + \ldots\ldots ,$$

$$b) \qquad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots ,$$

$$c) \qquad x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots ,$$

$$d) \qquad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \ldots ,$$

$$e) \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \ldots ,$$

$$f) \qquad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \ldots .$$

3. Show that the two series,

$$a) \qquad x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\,\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\,\frac{x^7}{7} + \ldots \ldots ,$$

$$b) \qquad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \ldots ,$$

are *convergent if* $-1 < x < 1$; *and that the second is convergent also if* $x = \pm 1$.

4. Show that the two series below are *convergent when* $-1 < x < 1$:

$$a) \qquad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \ldots ,$$

$$b) \qquad -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \ldots .$$

Will either of these two series be convergent when $x = -1$; when $x = 1$?

50. POWER SERIES.

The series,

$$(1) \qquad a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \ldots$$

containing only positive integral powers of x arranged in ascending order, in which the coefficients, a_0, a_1, a_2, etc, are either constants or independent of x, is called **a power series in** x.

A power series may be convergent for all finite values of its variable, as the series a) to f) in ex. 2 of § 49; or convergent for a limited interval of values of its variable, as the series in exs. 1, 3 and 4 of § 49.

I. *If series* (1) *be differentiated, term by term, we get*

$$(2) \qquad a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots \ldots .$$

If series (1) *be anti-differentiated, term by term, we get*

$$(3) \qquad k + a_0 x + \frac{a_1}{2} x^2 + \frac{a^2}{3} x^3 + \frac{a_3}{4} x^4 + \ldots \ldots .$$

The test-ratios for series (1), (2) and (3), are

$$\frac{a_{n+1}}{a_n}x,\quad \frac{n+1}{n}\,\frac{a_{n+1}}{a_n}x \quad\text{and}\quad \frac{n}{n+1}\,\frac{a_{n+1}}{a_n}x.$$

Let $\lim\limits_{n=\infty}\dfrac{a_{n+1}}{a_n}=l$, *when this limit exists.* Then, it is obvious that each of the three test-ratios approaches the limit lx, when n is infinite. Hence the three series (1), (2) and (3) are all convergent when $-1<lx<1$, or, when $l\gtrless 0$, $-\frac{1}{l}<x<\frac{1}{l}$; that is, *they are convergent for the same interval of values of x.*

II. If the power series,

(4) $$a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ ,$$

is convergent throughout the interval (a, b), *it may be either anti-differentiated, or integrated, term by term*, and the result will be valid for all values of x between a and b, $a<x<b$; also, *the series may be differentiated, term by term*, for all values of x between a and b, $a<x<b$.

III. If a power series,

(5) $$a_0+a_1x+a_2x^2+a_3a^3+a_4x^4+\ldots\ldots\ ,$$

vanishes for every value of x throughout a finite interval (a, b), which includes the value zero, $a<0<b$, *then each coefficient vanishes;* or,

$$a_0=a_1=a_2=\ldots\ldots=0.$$

IV. If two power series,

(6) $$a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ ,$$

(7) $$b_0+b_1x+b_2x^2+b_3x^3+\ldots\ldots\ ,$$

are equal for every value of x throughout a finite interval (a, b), including the value zero, $a<0<b$, *then coefficients of like powers are equal; that is,*

$$a_0=b_0,\quad a_1=b_1,\quad a_2=b_2,\ \ldots\ldots\ .$$

Exercises.

1. From the two series,

$$a) \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots ,$$

$$b) \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots ,$$

which are readily obtained by division, derive the following by multiplying *a*) and *b*) by dx, then integrating term by term:

$$c) \quad \int_0^x \frac{dx}{1+x} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots ,$$

$$d) \quad \int_0^x \frac{dx}{1-x} = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots .$$

2. Derive the following series from *a*) and *b*) by differentiation:

$$e) \quad \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots ,$$

$$f) \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots .$$

Verify them by division.

3. For what values of x are series *a*) to *f*) convergent? Which are convergent at $x = -1$; at $x = 1$?

CHAPTER XIX.

DEVELOPMENT OF FUNCTIONS IN SERIES: TAYLOR'S THEOREM AND SERIES.

51. THE DEVELOPMENT OF CERTAIN FUNCTIONS IN SERIES: TAYLOR'S THEOREM AND SERIES.

In exs. 1*a* and 1*b* of § 50 the student will observe the expansion of the functions, $(1+x)^{-1}$ and $(1-x)^{-1}$, in series, by ordinary division: in exs. 1*c* and 1*d* he will notice the series representing the functions, $\log(1+x)$ and $\log(1-x)$; these being obtained by integration. No doubt the student has already become acquainted with the fact that it is possible to develop (or, expand) some functions in series; and has asked himself whether there is any general method, or formula, by which functions might be developed, when they are capable of development. He has probably been told that logarithms, trigonometric functions, and other very useful tables of numbers, are computed by the aid of certain converging series; and has wanted to know how it could be done.

We shall now state, and prove, the general formula by which all functions which satisfy certain conditions may be expanded in power series; either into infinite series or finite sums, as the case may be.

Theorem. *Let $f(x)$ be a function satisfying the following conditions:*

α) $f(x)$ and its first $n-1$ derivatives, $f'(x), f''(x), f'''(x), \ldots\ldots f^{n-1}(x)$, are continuous throughout the interval (a, b);

β) the n^{th} derivative $f^n(x)$, is continuous for every value of x between a and b;

then, if $x \geqq a$, and $x+h \leqq b$,

$$\text{I.}\quad f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{h^n}{n!}f^n(x+\theta h),$$

where $0 < \theta < 1$.

This formula is known as **Taylor's Theorem.** For the special cases when $n = 1$, 2, and 3, the formula gives:

II. $f(x+h) = f(x) + hf'(x+\theta h)$;

III. $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x+\theta h)$;

IV. $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \dfrac{h^3}{3!}f'''(x+\theta h)$.

The term $\dfrac{h^n}{n!}f^n(x+\theta h)$ is sometimes called the **remainder**, and denoted by the symbol R_n.

If II is compared with (5), § 44, it will be evident that the *Law of the mean* is a special case of Taylor's Theorem.

Proof of Taylor's Theorem. Consider the following function of x:

$$(1)\quad F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) - A(b-x)^n,$$

which is made up from $f(x)$ and its first $n-1$ derivatives; and, in which A is an arbitrary constant.

If $x = b$, $F(x)$ vanishes. It may be seen that $F(x)$ is continuous throughout the interval (a, b); for $(b-x)^m$, where m is any positive integer, is continuous for the interval (a, b); and each of the other factors, $f(x), f'(x), \dots f^{n-1}(x)$, in the several terms, is continuous for the interval (a, b), by hypothesis.

The constant, A, since it may be any constant you please, may be so chosen that $F(x)$ shall vanish when we put $x = a$: then we should have,

$$(2)\quad F(a) = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) - A_1(b-a)^n = 0;$$

if we take

$$A_1 = \frac{1}{(b-a)^n}\left[f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)\right].$$

Give to A, in (1), the value A_1 thus found, and we shall have

$$F(a) = 0 \text{ and } F(b) = 0.$$

Hence, by Rolle's Theorem, § 43, the derivative of $F(x)$ must vanish for some value of x within the interval (a, b).

Differentiating (1), and putting $A = A_1$ in the result, we readily see that all the terms of the right member will vanish by cancellation, except the last two; that is,

$$(3) \qquad F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^n(x) + nA_1(b-x)^{n-1}$$

$$= (b-x)^{n-1}\left[nA_1 - \frac{1}{(n-1)!} f^n(x)\right].$$

Since this result must vanish for some value, x_1, between a and b; and since the factor $(b-x)^{n-1}$ cannot vanish when x is between a and b; we must have

$$(4) \qquad nA_1 - \frac{f^n(x_1)}{(n-1)!} = 0\,; \quad \text{or,} \quad A_1 = \frac{f^n(x_1)}{n!}.$$

where $a < x_1 < b$.

Now equate the values of A_1 as obtained from eqns. (2) and (4); and we get

$$(5) \qquad \frac{1}{(b-a)^n}\left[f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \right.$$

$$\left. \ldots - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)\right] = \frac{f^n(x_1)}{n!}.$$

Multiplying (5) by $(b-a)^n$ and transposing, we get the following:

$$(6) \quad f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) +$$

$$\frac{(b-a)^3}{3!}f'''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) +$$

$$\frac{(b-a)^n}{n!}f^n(x_1), \qquad a < x_1 < b.$$

Form (6) is obtained on the hypothesis that $f(x)$, etc., [see conditions α) and β)] satisfy certain conditions for the interval (a, b). Suppose, now, that x belongs to the interval (a, b); then the interval (a, x) will be contained in the interval (a, b); and conditions α) and β) will be fulfilled for the new interval (a, x), $a < x \leqq b$; so that we should get from (6), on replacing b by x, the form:

$$(7)\quad f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n}{n!}f^n(x_1), \qquad a < x_1 < x \leqq b.$$

Again, the interval (x, b), $a \leqq x < b$, is contained in the interval (a, b): therefore result (6) holds true for this interval (x, b); and, replacing a by x in (6) gives the form,

$$(8)\quad f(b) = f(x) + (b-x)f'(x) + \frac{(b-x)^2}{2!}f''(x) + \frac{(b-x)^3}{3!}f'''(x) + \dots + \frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{(b-x)^n}{n!}f^n(x_1), \quad a \leqq x < x_1 < b.$$

Finally, we may represent the difference, $b - x$, in (8) by h; that is, $b - x = h$. Then $b = x + h$, and $x < x_1 < x + h$: and it follows from this inequality that $x_1 = x + \theta h$ where $\theta =$ some proper fraction; that is, $0 < \theta < 1$.

Substituting these values in (8) we get theorem I, *which was to be proved.*

Forms (7) and (8) are merely different forms of I.

If $f(x)$ is such a function that conditions α) and β) are fulfilled *when n is indefinitely increased*, then we get from I the *series*,

$$(9)\quad f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots.$$

If $x = x_0$ is a fixed value of x, (9) becomes

$$(10)\quad f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots.$$

Series (10) is a *power series in h;* and its convergence may be tested as shown in § 49; or it may be tested by examining the limit of the remainder

$$R_n = \frac{h^n}{n!}f^n(x_0 + \theta h), \text{ when } n = \infty.$$

If $\lim\limits_{n=\infty} R_n = 0$, the series is convergent.

Series (9) is called *Taylor's Series.* Two other forms of the series may be obtained from (7) and (8). Each form has its advantages for different problems.

Exercises.

1. From $f(x) = \cos x$, obtain $f(x+h) = \cos(x+h)$ in the form of a series.

Solution. If $f(x) = \cos x$, then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f''''(x) = \cos x$, etc. It is obvious that the next four derivatives will be a repetition of the first four; and that the succeeding derivatives will only repeat, in endless succession, the same four values. Now, both $\sin x$ and $\cos x$ are continuous for all finite values of x; hence $f(x) = \cos x$ and all its derivatives satisfy conditions α) and β) for development by Taylor's series. Substituting in I we get

$$(a)\qquad \cos(x+h) = \cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \frac{h^4}{4!}\cos x - \frac{h^5}{5!}\sin x - \frac{h^6}{6!}\cos x + \ldots\ldots$$

$$= \cos x\left[1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \ldots\ldots\right] - \sin x\left[h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \ldots\ldots\right].$$

The two series in the square brackets are convergent for all finite values of h [see exs. 2*e* and 2*f*, § 49]; hence this series for $\cos(x+h)$ is convergent for all finite values of x, as well as for all finite values of h.

2. From $f(x) = \sin x$, obtain $\sin(x+h)$ in the form of a series. Show that it is convergent for all finite values of x and of h.

$$(a)\qquad \sin(x+h) = \sin x\left[1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \ldots\ldots\right] + \cos x\left[h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!} + \ldots\ldots\right].$$

3. From $f(x) = x^8$, obtain the expansion of $(x+h)^8$. Show in this case that there cannot be more than 9 terms in the expansion.

4. From $f(x) = e^x$, obtain e^{x+h} in the form of a series. Show that it is convergent for all finite values of x and h.

$$(a)\qquad e^{x+h} = e^x\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \ldots\ldots\right].$$

5. From $f(x) = \log_e x$, obtain $\log_e(x+h)$ in the form of a series.

[NOTE. In this example both $x + h > 0$ and $x > 0$].

$$(a)\qquad \log_e(x+h) = \log_e x + \frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3} - \frac{1}{4}\frac{h^4}{x^4} + \ldots\ldots$$

CHAPTER XX.

THE BINOMIAL THEOREM AND SERIES.

52. THE BINOMIAL THEOREM AND SERIES.

The binomial rule for finding the development of $(x+h)^m$ is rigorously proved in elementary Algebra only for the case of m *a positive integer.* By means of Taylor's Theorem and Series it may be established when m *is any constant*, provided x and h are so chosen that the resulting series shall be convergent.

Let $f(x)=x^m$, *where* m *is any constant; it is required to find* $f(x+h)=(x+h)^m$.

The successive derivatives of x^m are

$$f'(x)=mx^{m-1}, \qquad f''(x)=m(m-1)x^{m-2},$$
$$f'''(x)=m(m-1)(m-2)x^{m-3},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(1) $f^n(x)=m(m-1)(m-2)\ldots\ldots(m-n+1)x^{m-n};$
where n *is always a positive integer.*

Case I. *Suppose* m *a positive integer.*

In this case n will ultimately equal m, in (1); and we get, when $n=m$,

(2) $f^n(x)=f^m(x)=m(m-1)(m-2)\ldots\ldots 3\cdot 2\cdot 1=m!$

Whence $f^{m+1}(x)=0$, and all higher derivatives vanish. The differentiation closes with the m^{th} derivative; and both x^m and its m derivatives, are continuous for all finite values of x. Substituting in Taylor's Theorem, § 51, I, we get

$$(3) \quad (x+h)^m=x^m+mx^{m-1}h+\frac{m(m-1)}{2!}x^{m-2}h^2+ \frac{m(m-1)(m-2)}{3!}x^{m-3}h^3+\ldots\ldots+\frac{m!}{m!}h^m.$$

This we shall call the **Binomial Theorem.** It is the ordinary algebraic form; and is valid for all finite values of x and h, and for all positive integral values of m. It contains $m+1$ terms.

Case II. *Suppose m is not a positive integer.*

In this case n and m, in equation (1), can never be equal; so that the differentiation of x^m can be continued indefinitely; or, the derivatives are unlimited in number.

Now, let an interval $(x, x+h)$ be so chosen that x^m and all its derivatives, are continuous in this interval: then we get, from Taylor's Series, § 51, (9),

$$(4) \quad (x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{2!}x^{m-2}h^2 + \frac{m(m-1)(m-2)}{3!}x^{m-3}h^3 + \ldots\ldots .$$

We shall call this **The Binomial Series.** It is a power series with respect to h. Finding its test-ratio we get

$$(5) \quad \lim_{n=\infty}\left[\frac{u_{n+1}}{u_n}\right] = \lim_{n=\infty}\left[\frac{m-n}{n+1}\cdot\frac{h}{x}\right] = \frac{h}{x}\lim_{n=\infty}\left[\frac{\frac{m}{n}-1}{1+\frac{1}{n}}\right] = -\frac{h}{x}.$$

The negative sign in the result (5) shows that, when h and x have the same sign, series (4) is alternating after the term in which $n > m$; and that when h and x have opposite signs, all the terms of (4) will have the same sign after the term in which $n > m$.

A) *Suppose $h < x$, or $h > x$.*

Since $\lim\limits_{n=\infty}\left[\frac{u_{n+1}}{u_n}\right] = -\frac{h}{x}$, [see § 49, IX,]

series (4) will be *convergent when $h < x$*, and *divergent when $h > x$.*

B) *Suppose $h = -x$.*

In this case $\lim\limits_{n=\infty}\left[\frac{u_{n+1}}{u_n}\right] = 1$: and we must use test VI, of § 49. We get

$$^*(6) \quad \lim_{n=\infty}\left[n\left(1-\frac{u_{n+1}}{u_n}\right)\right] = \lim_{n=\infty}\left[\frac{n}{n+1}(m+1)\right] = m+1;$$

which shows that, when $h = -x$, series (4) is *convergent if $m+1 > 1$, or $m > 0$*; and *divergent if $m+1 < 1$, or $m < 0$.*

The series is, obviously, *convergent if $m = 0$.*

---

* **Test VI applies, since we have shown that when $h = -x$ the terms of series (4) will all have the same sign after $n > m$.**

C) Suppose $h = x$.

If $h = x$ series (4) is alternating. It will be convergent, by test VII, § 49, *when the terms ultimately decrease continually in numerical value, and approach the limit zero.*

We find the terms u_{n-1} and u_n to be,

$$(7)\quad u_{n-1} = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot \cdots\cdots \frac{m+2-n}{n-1} \cdot x^m,$$

$$(8)\quad u_n = \frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdots\cdots \frac{m+2-n}{n-1} \cdot \frac{m+1-n}{n} \cdot x^m;$$

whence

$$(9)\qquad u_n = \frac{m+1-n}{n} u_{n-1}.$$

From (9) it is evident that: —

a) If $m < -1$ the numerical value of $\frac{m+1-n}{n}$ will be greater than unity for every value of n (n being a positive integer); and that the term u_n is, therefore, numerically greater than the term u_{n-1} always; that is, the terms of series (4) increase continually. Hence, the *series must be divergent if* $m < -1$.

β) If $m = -1$, then $u_n = -u_{n-1}$: hence, the terms of series (4) will be numerically equal, but opposite in sign. [Such a series is called an *oscillating* series.]

In this case (4) *is divergent.*

γ) If $m > -1$, $\frac{m+1-n}{n}$ will be numerically less than unity when $n > \frac{m+1}{2}$: hence, provided $m > -1$, $u_n < u_{n-1}$ for all values of $n > \frac{m+1}{2}$; and the first condition of test VII, § 49, is satisfied. To show that the second condition is also satisfied, viz. $\lim\limits_{n=\infty} u_n = 0$, we may put (8) in the following form:

$$(10)\quad u_n = (-1)^n \frac{-m}{1} \cdot \frac{1-m}{2} \cdot \frac{2-m}{3} \cdot \frac{3-m}{4} \cdot \cdots\cdots \frac{n-(m+1)}{n} x^m$$

$$= (-1)^n \left(1 - \frac{m+1}{1}\right)\left(1 - \frac{m+1}{2}\right)\left(1 - \frac{m+1}{3}\right) \cdots\cdots \left(1 - \frac{m+1}{n}\right) x^m.$$

Since $m > -1$, $m + 1 > 0$; and, by X, § 49, the infinite product in the right member of (10) converges to the limit zero when $n = \infty$; hence, $\lim_{n=\infty} u_n = 0$, and condition (β) of test VII, § 49 is satisfied: and *series* (4) *is convergent when* $h = x$ *and* $m > -1$.

To sum up: we have shown that the binomial series,

$$(11)\quad (x + h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{2!}x^{m-2}h^2 + \frac{m(m-1)(m-2)}{3!}x^{m-3}h^3 + \ldots ,$$

a) *is divergent if h is numerically greater than x;*

b) *is convergent if h is numerically less than x;*

c) *is convergent if $h = -x$ and $m \geqq 0$;*

d) *is divergent if $h = -x$ and $m < 0$;*

e) *is divergent if $h = x$ and $m \leqq -1$;*

f) *is convergent if $h = x$ and $m > -1$.*

It will be evident upon examining the foregoing, that h and x may be interchanged from beginning to end of the demonstrations; that is. by regarding h as variable, we might have treated the function $f(h) = h^m$ for the interval $(h, h + x)$, and have obtained a binomial series for $f(h + x) = (h + x)^m$, arranged in ascending powers of x instead of h. We should have then

$$(12)\quad (h + x)^m = h^m + mh^{m-1}x + \frac{m(m-1)}{2!}h^{m-2}x^2 + \frac{m(m-1)(m-2)}{3!}h^{m-3}x^3 + \ldots .$$

The conditions of convergence, *a*) to *f*), above, will apply to series (12) if we interchange x and h in them.

It follows, then, that $(x + h)^m$, *when m is not a positive integer, can be developed in a converging power series for all cases when h and x are not equal.* For example, if $h < x$ we should develop in the form $(x + h)^m = x^m + m\,x^{m-1}h +$ etc., by (4): but, if $x < h$, it may be developed in the form $(h + x)^m = h^m + m\,h^{m-1}x +$ etc., by (12).

A very useful form of the binomial series is obtained by putting $h = 1$ in (12). We get:

$$(13) \qquad (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots .$$

Series (13) will be *divergent if x is numerically greater than unity; convergent if x is numerically less than unity; convergent or divergent when $x = 1$ according as $m > -1$ or $m \leqq -1$; convergent or divergent when $x = -1$ according as $m \geqq 0$ or $m < 0$.*

Exercises.

1. Obtain the following series:

a) $(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1 \cdot 1}{2 \cdot 4}x^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^6 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^8 - \ldots ,$

[*convergent if* $-1 \leqq x \leqq 1$];

b) $(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \ldots ,$

[*convergent if* $-1 < x < 1$];

c) $(x^2-1)^{\frac{1}{2}} = x - \frac{1}{2}x^{-1} - \frac{1 \cdot 1}{2 \cdot 4}x^{-3} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^{-5} - \ldots ,$

[*convergent if* $+1 \leqq x \leqq -1$];

d) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots ,$

[*convergent if* $-1 < x < 1$].

2. Obtain the series for the integrals given below from the proper series of Ex. 1:

a) $\int_0^x (1-x^2)^{\frac{1}{2}}\,dx = x - \frac{1}{2}\frac{x^3}{3} - \frac{1 \cdot 1}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{x^7}{7} - \ldots ,$

[*convergent if* $-1 \leqq x \leqq 1$];

b) $\int_0^x (1-x^2)^{-\frac{1}{2}}\,dx = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots ,$

[*convergent if* $-1 < x < 1$];

c) $\int_0^x (x^2-1)^{\frac{1}{2}}\,dx = \frac{x^2}{2} - \frac{1}{2}\log x + \frac{1 \cdot 1}{2 \cdot 4}\cdot\frac{1}{2x^2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\cdot\frac{1}{4x^4} + \ldots$

[*convergent if* $x > 1$].

CHAPTER XXI.

MACLAURIN'S THEOREM AND SERIES.

53. MACLAURIN'S THEOREM AND SERIES.

This theorem is merely an important special case of Taylor's theorem. Form (7) of § 51 will furnish the expansion of $f(x)$ for an interval (a, x) when $f(x)$ and its first n derivatives satisfy the conditions α) and β) at the beginning of § 51. Now, a is a constant, and it may have the value zero; that is: —

If $f(x)$ and its first n derivatives satisfy the conditions α) and β) of Taylor's theorem in the interval $(0, x)$, then $f(x)$ may be expanded by (7), which takes the form,

$$*(1)\quad f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(0) + \dots\dots$$
$$+ \frac{x^{n-1}}{(n-1)\,!}f^{n-1}(0) + \frac{x^n}{n\,!}f^n(\theta x),\ 0 < \theta < 1,$$

where $x_1 = \theta x$, or $0 < \theta x < x$.

Form (1) is known as **Maclaurin's Theorem.** The term $\frac{x^n}{n\,!}f^n(\theta x)$ is called the remainder: it is the $(n+1)^{th}$ term, and may be denoted by R_n.

If $f(x)$ and its first n derivatives remain continuous in the interval $(0, x)$ when n is indefinitely increased, we get from (1) the series,

$$(2)\quad f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(0) + \frac{x^4}{4\,!}f''''(0) + \dots$$

This is **Maclaurin's Series.** It is a power series, and it may be shown to be convergent or divergent, either by the tests of § 49, or

---

* The symbol $f\ (0)$ represents the value of $f(x)$ when $x = 0$;
" " $f'(0)$ " " $f'(x)$ " $x = 0$;
" " $f'\ (0)$ " " $f''(x)$ " $x = 0$;
. . . .

by examining $\lim\limits_{n=\infty} R_n$. *It will be convergent if* $\lim\limits_{n=\infty} R_n = 0$; for, then, the error caused in taking the sum of the first n terms to represent the value of the series, can be made arbitrarily small by taking n sufficiently great.

Example.

Develop $f(x) = a^x$ into its power series.

We get readily, $f'(x) = a^x \log a$, $f''(x) = a^x \log^2 a$, $f'''(x) = a^x \log^3 a \ldots$; whence, $f(0) = a^0 = 1$, $f'(0) = \log a$, $f''(0) = \log^2 a$, $f'''(0) = \log^3 a \ldots$.

Substituting in Maclaurin's series, we get:

$$(3) \quad a^x = 1 + x \log a + \frac{x^2 \log^2 a}{2!} + \frac{x^3 \log^3 a}{3!} + \frac{x^4 \log^4 a}{4!} + \ldots .$$

If $a = e = 2.71828 \ldots$, $\log a = 1$; and (3) gives:

$$(4) \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots .$$

Series (3) *and* (4) *are convergent for all finite values of* x.

Exercises.

Develop the following functions into their power series as given: —

1. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$,

   [*Convergent if* $-\infty < x < \infty$];

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$,

   [*Convergent if* $-\infty < x < \infty$];

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$,

   [*Convergent if* $-\infty < x < \infty$];

4. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots$,

   [*Convergent if* $-1 < x \leqq 1$];

5. $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$,

   [*Convergent if* $-1 \leqq x < 1$];

6. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots$,

   [*Convergent if* $-1 < x < 1$];

7. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$,

[*Convergent if* $-1 \leqq x \leqq 1$];

8. $e^{kx} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \dots$, $k =$ *any constant*,

[*Convergent* $-\infty < x < \infty$];

9. $e^{-kx} = 1 - kx + \frac{k^2x^2}{2!} - \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} - \dots$, $k =$ *any constant*,

[*Convergent if* $-\infty < x < \infty$];

10. $\sin kx = kx - \frac{k^3x^3}{3!} + \frac{k^5x^5}{5!} - \frac{k^7x^7}{7!} + \dots$, $k =$ *any constant*,

[*Convergent if* $-\infty < x < \infty$];

11. $\cos kx = 1 - \frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} - \frac{k^6x^6}{6!} + \dots$, $k =$ *any constant*,

[*Convergent if* $-\infty < x < \infty$].

12. Change k to $k\sqrt{-1} = ki$ in Exs. 8 and 9; reduce the results by using the formulæ $i^{4m} = 1$, $i^{4m+1} = i$, $i^{4m+2} = -1$, and $i^{4m+3} = -i$, $m =$ any integer; then, substitute from Exs. 10 and 11; and we get:

a) $e^{ikx} = \cos kx + i\sin kx$; *b*) $e^{-ikx} = \cos kx - i\sin kx$.

If $k = 1$, *a*) and *b*) become:

c) $e^{ix} = \cos x + i\sin x$; *d*) $e^{-ix} = \cos x - i\sin x$.

* From *c*) and *d*) we get:

e) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$; *f*) $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.

13. The expressions, $\frac{e^x - e^{-x}}{2}$ and $\frac{e^x + e^{-x}}{2}$, are called, respectively, the hyperbolic sine and hyperbolic cosine of x; and are abbreviated to *sinh x* and *cosh x*.

† Show that:

a) $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$;

b) $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$.

[*Both convergent if* $-\infty < x < \infty$.]

---

* Expressions *e*) and *f*) are exponential values of the circular sine and cosine of x. Compare them with the exponential values of the hyperbolic sine and cosine of x defined in Ex. 13.

† Compare *a*) and *b*) with Exs. 2 and 3 above.

14. Using a), b), c), and d), of ex. 12, prove the following:—

a) $(\cos x + i\sin x)^k = (e^{ix})^k = e^{ikx} = \cos kx + i\sin kx$;

b) $(\cos x - i\sin x)^k = (e^{-ix})^k = e^{-ikx} = \cos kx - i\sin kx$.

These results are known as **De Moivre's Theorems.** They may be used in calculating the roots of numbers. For example, we may find the three cube roots of unity as follows:

$$1 = \cos 2k\pi + i\sin 2k\pi\ , \qquad k = 1, 2, 3, \ldots .$$

$$\therefore \sqrt[3]{1} = [\cos 2k\pi + i\sin 2k\pi]^{\frac{1}{3}}$$

$$= \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}, \qquad \text{by Ex. 14, } a.$$

If $k = 1$ we get,

$$\sqrt[3]{1} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \cos 120^\circ + i\sin 120^\circ = -\frac{1}{2} + \frac{i}{2}\sqrt{3}.$$

If $k = 2$ we get,

$$\sqrt[3]{1} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos 240^\circ + i\sin 240^\circ = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

If $k = 3$ we get,

$$\sqrt[3]{1} = \cos 2\pi + i\sin 2\pi = \cos 360^\circ + i\sin 360^\circ = 1.$$

Putting $k = 4, 5, 6$, and so on, will not give any new value of $\sqrt[3]{1}$; but will repeat the three already found.

15. Find the four fourth roots, and the five fifth roots, and the six sixth roots of 1, by using Ex. 14, a).

16. Find the three cube roots, and the four fourth roots, of -1, by using Ex. 14, a), having given,

$$-1 = \cos k\pi + i\sin k\pi\,, \quad k = 1, 3, 5, 7, \ldots .$$

17. Find

$$\int\frac{\sin x\,dx}{x}, \quad \int\frac{\sin x\,dx}{x^2}, \quad \int\frac{\cos x\,dx}{x} \text{ and } \int\frac{\cos x\,dx}{x^2},$$

from Exs. 2 and 3.

CHAPTER XXII.

COMPUTATION OF NUMERICAL TABLES BY MEANS OF SERIES.

54. COMPUTATION BY MEANS OF SERIES.

A divergent series can not be used for computing. Convergent series alone may be so used; and it is desirable that the series should *converge rapidly*, — that the * *sum of the series* should be obtainable correctly to four or five decimal places from a small number of the first terms of the series. When a large number of terms have to be reckoned in order to secure a close approximation to the sum of the series, it is said to *converge slowly*, or, to be *slowly convergent*.

A. Calculation of base, e.

In the series,

(1) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad \text{[See (4), § 53]},$$

put $x = 1$, and we get,

(2) $$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots\dots .$$

Reckoning the first sixteen terms of (2) we get

$$e = 2.71828\ 18284\ 59 \dots\dots .$$

Exercises.

1. Find from series (1):

a) $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots\dots = 7.38905 \dots\dots ;$

b) $\sqrt{e} = 1 + \frac{1}{2} + \frac{1}{2^2 2!} + \frac{1}{2^3 3!} + \dots\dots = 1.64872 \dots\dots .$

2. Compute the hyperbolic sine and cosine of unity from the series in Exs. 18 *a*) and 18 *b*) of § 53.

Ans. $\sinh 1 = 1.1752 \cdots\cdots ; \quad \cosh 1 = 1.5431 \cdots\cdots .$

---

* By *sum of the series* we mean: *the limit of the sum of the first n terms as n is indefinitely increased.* Only convergent series have a sum.

B. Calculation of π.

Putting $x = \frac{1}{2}$ in the series for $\sin^{-1}x$ (see ex. 6, § 53) we get:

$$(3)\quad \frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \frac{1}{2} + \frac{1}{2}\,\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5}\left(\frac{1}{2}\right)^5 + \cdots\;.$$

Series (3) converges rather slowly. A much more rapidly converging series may be found by using the series for $\tan^{-1}x$ (see Ex. 7, § 53) as a basis, and deriving a new series, as follows:—

Putting, successively, $x = \frac{1}{5}$ and $x = \frac{1}{239}$ in the series for $\tan^{-1}x$, we get:

$$(4)\quad \alpha = \tan^{-1}\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \cdots\;;$$

$$(5)\quad \beta = \tan^{-1}\frac{1}{239} = \frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \cdots\;.$$

Since $\tan\alpha = \frac{1}{5}$ and $\tan\beta = \frac{1}{239}$, we may obtain by trigonometric reduction,

$$(6)\quad \tan(4\alpha - \beta) = \frac{\tan 4\alpha - \tan\beta}{1 + \tan 4\alpha\tan\beta} = 1\,.$$

Whence, $4\alpha - \beta = \tan^{-1}1 = \frac{\pi}{4}$; and we find, from (4) and (5),

$$(7)\quad \frac{\pi}{4} = 4\alpha - \beta = 4\left[\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \cdots\right] - \left[\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \cdots\right].$$

$$(8)\quad \therefore\quad \pi = \frac{16}{5}\left[1 - \frac{4}{3\cdot 100} + \frac{4^2}{5\cdot 100^2} - \frac{4^3}{7\cdot 100^3} + \cdots\right] - \frac{4}{239}\left[1 - \frac{1}{3(57121)} + \frac{1}{5(57121)^2} - \cdots\right].$$

Exercise.

1. Find π to three places of decimals, using, first, series (8) and then series (3) so as to compare the rapidity of convergence in the two series.

*** *Ans.* $\pi = 3.14159\ 26535\ 89793\ \ldots\;.$**

*** The result is given to fifteen places for reference merely.**

C. Calculation of Sines and Cosines.

In the series for $\sin x$ and $\cos x$, of Exs. 2 and 3, § 53, put $x = \frac{k\pi}{2}$; and we get:

$$(9)\quad \sin\frac{k\pi}{2} = k\frac{\pi}{2} - \frac{k^3}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{k^5}{5!}\left(\frac{\pi}{2}\right)^5 - \cdots\cdots ;$$

$$(10)\quad \cos\frac{k\pi}{2} = 1 - \frac{k^2}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{k^4}{4!}\left(\frac{\pi}{2}\right)^4 - \cdots\cdots .$$

The numerical values of the coefficients of the powers of k can be computed, once for all, and substituted in (9) and (10). We thus get:

$$(11)\qquad \begin{aligned}\sin\frac{k\pi}{2} &= 1.57079\ 633\,k - 0.64596\ 410\,k^3 \\ &+ 0.07969\ 263\,k^5 - 0.00468\ 175\,k^7 \\ &+ 0.00016\ 044\,k^9 - 0.00000\ 360\,k^{11} \\ &+ 0.00000\ 006\,k^{13} - \cdots\cdots ;\end{aligned}$$

$$(12)\qquad \begin{aligned}\cos\frac{k\pi}{2} &= 1.00000\ 000 - 1.23370\ 055\,k^2 \\ &+ 0.25366\ 951\,k^4 - 0.02086\ 348\,k^6 \\ &+ 0.00091\ 926\,k^8 - 0.00002\ 520\,k^{10} \\ &+ 0.00000\ 047\,k^{12} - 0.00000\ 001\,k^{14} \\ &+ \cdots\cdots .\end{aligned}$$

Since it is not necessary to compute the functions of angles larger than 45°, or $\frac{\pi}{4}$, the k in (11) and (12) needs to be taken only from 0 to $\frac{1}{2}$.

If $\frac{k\pi}{2} = \frac{\pi}{180} = 1°$, $k = \frac{1}{90}$. If $\frac{k\pi}{2} = \frac{\pi}{10800} = 1'$, $k = \frac{1}{5400}$.
If $\frac{k\pi}{2} = \frac{\pi}{60\cdot 60\cdot 180} = 1''$, $k = \frac{1}{324000}$.

In constructing a table of natural sines and cosines for angles differing by 10′, we might proceed as follows: —

If $\frac{k\pi}{2} = 10'$, $k = \frac{1}{540}$. Substituting $k = \frac{1}{540}$ in (11) and (12) we find,

$$\sin 10' = 0.00290\ 888, \qquad \cos 10' = 0.99999\ 577.$$

Then, using the trig. formulae for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, we get,

$$\sin 20' = 2 \sin 10' \cos 10', \qquad \cos 20' = 1 - 2\sin^2 10'.$$

In like manner,

$$\sin 30' = \sin(20' + 10') = \sin 20' \cos 10' + \cos 20' \sin 10',$$
$$\cos 30' = \cos(20' + 10') = \cos 20' \cos 10' - \sin 20' \sin 10' \quad .$$

So, step by step, the entire table could be formed.

The other trigonometric functions may be computed from the sines and cosines.

Exercises.

1. Compute the following by using series (11) and (12).

a)	$\sin 5'\ 10'' = 0.00150292$;	*c*)	$\sin 18^\circ = 0.30901700$;
b)	$\cos 5'\ 10'' = 0.9999988$;	*d*)	$\cos 18^\circ = 0.95105651$.

D. Napierian, or Natural, Logarithms.

The base of Napierian logarithms is the number, $e = 2.7182818 \ldots$, — called, simply, "base." Its value is obtained from the series (2) of this section.

A rapidly converging series for computing logarithms may be derived in the following manner: —

(13) * $\log \dfrac{1+x}{1-x} = \log(1+x) - \log(1-x)$, [by theory of logarithms].

Substituting in (13) the series for $\log(1+x)$ and $\log(1-x)$ [both of which are convergent if $-1 < x < 1$] from Exs. 4 and 5, § 53, we readily find,

$$(14) \quad \log \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right];$$

which is easily shown to be convergent if $-1 < x < 1$.

Now, put $$\frac{1+x}{1-x} = \frac{N}{N'}; \text{ whence, } x = \frac{N-N'}{N+N'}.$$

* When the base is not indicated, e is to be understood in this book.

An inspection of the result $x = \dfrac{N - N'}{N + N'}$ will show that *the condition,* $-1 < x < 1$, *will be satisfied if* N *and* N' *are any two positive numbers whatsoever.* Substituting in (14) we get:

$$(15)\quad \log \frac{N}{N'} = \log N - \log N' = 2\left[\frac{N - N'}{N + N'} + \frac{1}{3}\left(\frac{N - N'}{N + N'}\right)^3 + \frac{1}{5}\left(\frac{N - N'}{N + N'}\right)^5 + \dots\right].$$

Series (15) is convergent for all positive values of N and N': and it is always possible so to choose N and N' as to make it converge rapidly.

Putting $N = 2$ and $N' = 1$ we get (since $\log N' = \log 1 = 0$, and $\dfrac{N - N'}{N + N'} = \dfrac{1}{3}$);

$$(16)\quad \log N = \log 2 = 2\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \dots\right]$$
$$= 0.69314\ 718\ \dots\ .$$

Putting $N = 3$ and $N' = 2$, we get $\dfrac{N - N'}{N + N'} = \dfrac{1}{5}$; and, from (15),

$$(17)\quad \log 3 = \log 2 + 2\left[\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \dots\right]$$
$$= 1.09861\ 229\ \dots\ .$$

It is necessary to compute, by series, the logarithms of prime numbers only. For example,

$$\log 4 = \log 2^2 = 2 \log 2 = 1.38629\ 436;$$
$$\log 6 = \log (3 \times 2) = \log 3 + \log 2 = 1.79175\ 947.$$

Exercises.

1. Compute from series (15) log 5 and log 11.

Ans. $\log 5 = 1.60943791$; $\log 11 = 2.39789527$.

2. Compute log 24 and log 660.

Ans. $\log 24 = 3.17805383$; $\log 660 = 6.49223984$.

E. Briggs's, or Common, Logarithms.

The base of Common logarithms is the number 10. The relation between the Napierian and common logarithms may be shown as follows: —

Denote the common logarithm of the positive number N by x; that is, $\log_{10} N = x$. We get, then,

$$N = 10^x\,; \quad \text{and,} \qquad \therefore \quad \log_e N = \log_e 10^x = x \log_e 10\,.$$

Whence,

$$(18) \quad x = \frac{\log_e N}{\log_e 10} = \mathbf{M} \log_e N = \log_{10} N\,.$$

$$\text{The multiplier } \mathbf{M} = \frac{1}{\log_e 10} = 0.43429\,44819\;\ldots\;,$$

is called the **Modulus** of common logarithms.

Multiplying series (15) by $\mathbf{M}$, and transposing, we get,

$$(19)\ \log_{10} N = \log_{10} N' + 2\,\mathbf{M}\left[\frac{N-N'}{N+N'} + \frac{1}{3}\left(\frac{N-N'}{N+N'}\right)^3 + \ldots\right];$$

since $\mathbf{M} \log_e N = \log_{10} N$, and $\mathbf{M} \log_e N' = \log_{10} N'$, by (18).

Common logarithms may be computed either directly from series (19), or by using a table of Napierian logarithms.

Exercises.

1. Compute $\log_{10} 2 = 0.301030$ by both methods.

2. Given $\log 65 = 4.174387$, compute $\log_{10} 65 = 1.812913$.

3. Compute $\log_{10} 0.65 = \log_{10} \frac{65}{100} = \log_{10} 65 - \log_{10} 100 = \bar{1}.812913$ by putting $N = 65$ and $N' = 100$ in series (19).

CHAPTER XXIII.

DEFINITION OF THE LENGTH OF A PLANE CURVE: COMPUTATION OF LENGTHS, AND DIFFERENTIAL OF LENGTH, IN RECTANGULAR COÖRDINATES.

55. DEFINITION OF THE LENGTH OF A PLANE CURVE.

The **length of a straight line** *is the number that expresses how many times the line contains the linear unit.* This number must be found either by direct superposition of the linear unit — as when the carpenter measures a beam by end to end applications of his foot-rule; or by more complicated operations which are developed from superposition — as, in a continental survey, when a system of triangulation is based upon the exact measurement of one or more level lines by end to end applications of a straight measuring bar.

The length of a series of connected straight lines is the number that expresses the sum of the lengths of the lines. *For example, the perimeter of a polygon of n sides is the sum of the lengths of the sides.*

No straight line can be superposed upon a curved line, or curve; since no straight line can coincide with a curve, except at one or more isolated points. This fact makes it impossible to measure curves as straight lines are measured; and makes it impossible, therefore, to define "length" for a curve in the same way that length is defined for a straight line; *since a definition of "length of a plane curve" must be the basis for measuring the thing defined.*

To define "length of a plane curve" we must use the notion of a limit. For example, the *circumference of a circle* is defined as follows: — (1) The inscribed and circumscribed regular polygons of n sides are drawn about the circle; (2) it is shown that *if n increases*, the perimeter of the inscribed polygon is an increasing variable which remains always less than the perimeter of the circumscribed polygon; while the perimeter of the circumscribed polygon is a decreasing variable, always greater than the perimeter of the inscribed polygon;

(3) that the difference between two corresponding inscribed and circumscribed perimeters approaches the limit zero when n increases indefinitely, and, therefore, the perimeters approach a common limit; (4) and, finally, *this* **limit** *is defined as the length of the circumference of the circle.*

[See Chauvenet's Geometry, revised edition, Book V, Prop. VII.]

In an analogous way we may define the length of a given plane curve as follows: —

Take an arc whose end points are A and B. [See Fig. 27.] Let it be divided into n parts, which may be equal or not. Draw the n successive chords of the smaller arcs, joining A and B by a broken line. [We may call the figure formed by the chords, *an inscribed polygon of n sides;* although it will not be a closed polygon unless the points A and B coincide.] Then, let n be indefinitely increased in such a way that each of the n arcs is indefinitely diminished. The perimeter of the inscribed polygon has a definite value for every value of n. Moreover, it obviously varies with n. Then, we say that: — *If the perimeter of such an inscribed polygon approaches a limit, when n becomes infinite,* **this limit is the length of the curve.**

We shall prove in § 56 that, under certain conditions, this limit exists; and shall derive a general method for calculating it — a method based on anti-differentiation, as was the method of calculating areas given in § 33.

56. CALCULATION OF THE LENGTH OF A GIVEN CURVE.

Let the given curve be the locus of a function $y = f(x)$; which, together with its first derivative, $\frac{dy}{dx} = f'(x)$, is single-valued and continuous throughout the interval (a, b). It is required to find the length of the arc AB (see Fig. 27).

Let $OM_0 = a$, and $OM_n = b$, be the abscissæ of A and B. Divide the length $b - a$ into n equal parts, each equal to $\frac{b-a}{n} = \Delta x = MN$; and draw ordinates through the points of division. Draw the n chords AP_1, P_1P, PQ, etc., through the points of division of the arc AB. *These chords form an unclosed polygon, whose perimeter-limit, when $n = \infty$, is the length of the arc AB, by definition.*

Take any chord, PQ, whose end points are (x, y) and $(x + \Delta x, y + \Delta y)$. Then $PR = \Delta x$, $RQ = \Delta y$; and we have,

$$(1) \qquad PQ = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x\,.$$

Now, the ratio $\frac{\Delta y}{\Delta x}$ in (1) is the slope of the chord PQ; and this is [by the Law of the Mean, § 44, Ex. 3] equal to the value of the derivative, $f'(x)$, at some point between P and Q. Represent the value of x at this point by ξ; then,

$$(2) \qquad \frac{\Delta y}{\Delta x} = f'(\xi), \text{ where } x < \xi < x + \Delta x\,.$$

Remembering that $\Delta x = dx$ (§ 22), and substituting in (1), we get,

$$(3) \qquad PQ = \sqrt{1 + [f'(\xi)]^2}\,dx\,.$$

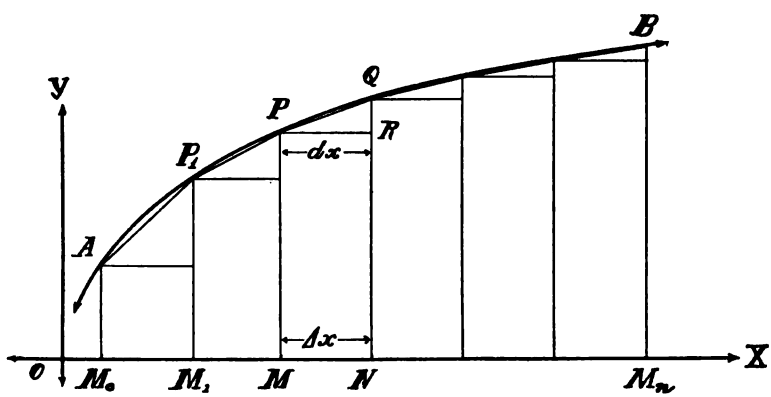

FIG. 27.

An expression of the same form as (3) can be found for the length of each of the n chords, or sides, of the inscribed polygon. The sum of the n expressions like (3) will represent the perimeter of the inscribed polygon from A to B. This sum may be denoted by ΣPQ. *Then we shall have, for the length of the arc AB,*

$$(4) \qquad \lim_{n=\infty} \sum_a^b PQ = \lim_{n=\infty} \sum_a^b \sqrt{1 + [f'(\xi)]^2}\,dx\,.$$

The summation expressed in the last member of (4) relates to the variable x: and x is to increase from a to b by taking, successively, the n values a, $a + dx$, $a + 2dx$, $a + 3dx$, ... , $b - dx$.

But ξ must take n successive values, $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$, such that $a < \xi_1 < a + dx < \xi_2 < a + 2dx < \xi_3 < a + 3dx < \ldots < \xi_{n-1} < b - dx < \xi_n < b$. This follows from the definition of ξ given above in (2). We must now prove that the ξ in (4) may be replaced by x: in other words, we wish to prove that,

$$(5) \qquad \lim_{n=\infty} \sum_a^b \sqrt{1 + [f'(\xi)]^2}\,dx = \lim_{n=\infty} \sum_a^b \sqrt{1 + [f'(x)]^2}\,dx,$$

when $x < \xi < x + dx$.

We may prove (5) as follows: — Let the curve $A'B'$ [fig. 28] be the locus of the function, $y = \phi(x) \equiv \sqrt{1 + [f'(x)]^2}$, from $OM_0 = a$ to $OM_n = b$. Construct $M_0M_1 = M_1M_2 = \ldots = \dfrac{b-a}{n} = \Delta x$: and let the abscissæ OM', OM'', etc., represent the successive values of ξ; each of which, by hypothesis, lies between two adjacent values of x. Then, the dotted ordinates, $M'P'$, $M''P''$, etc., will represent the successive corresponding values of $y = \phi(\xi) \equiv \sqrt{1 + [f'(\xi)]^2}$.

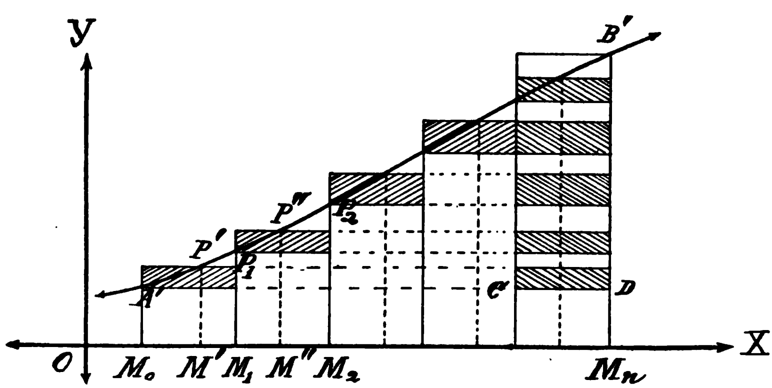

FIG. 28.

Construct the two sets of rectangles on the common bases, M_0M_1, M_1M_2, etc., as shown in the figure. Call the rectangles that overlap the curve, *the first set*. These have the dotted lines for altitudes. Call those inscribed under the curve, *the second set*. They have the

ordinates $M_0 A'$, $M_1 P_1$, etc., for altitudes. *Represent by* $\mathbf{V}$ *the sum of the areas of the first set; and by* $\mathbf{U}$ *the sum of the areas of the second set.*

The difference, $\mathbf{V}-\mathbf{U}$, is readily seen to be equal to the shaded portion of the rectangle CB'; and, therefore, obviously, $\mathbf{V}-\mathbf{U} <$ rectangle CB'. Also, $\mathbf{V} > \mathbf{U}$, or $\mathbf{V}-\mathbf{U} > 0$. Hence,

$$(6) \qquad 0 < \mathbf{V} - \mathbf{U} < \text{rectangle } CB'.$$

But the base $CD = \Delta x$, of the rectangle CB', diminishes indefinitely when $n = \infty$; so that $\lim\limits_{n=\infty} [\text{rectangle } CB'] = 0$.

Hence, we have $\lim\limits_{n=\infty} [\mathbf{V}-\mathbf{U}] = 0$, from (6); and, therefore,

$$(7) \qquad \lim_{n=\infty} \mathbf{V} = \lim_{n=\infty} \mathbf{U}.$$

But, $\qquad \mathbf{V} = \sum\limits_a^b \sqrt{1+[f'(\xi)]^2}\,dx$; and,

$\mathbf{U} = \sum\limits_a^b \sqrt{1+[f'(x)]^2}\,dx$: hence, by substituting for $\mathbf{V}$ and $\mathbf{U}$ in (7) we obtain (5), *which was to be proved.*

But it has been shown in § 33, that $\lim\limits_{n=\infty} \mathbf{U}$ is the area under the curve $A'B'$; and that this limit can be found by anti-differentiation: in short,

$$(8) \qquad \lim_{n=\infty} \mathbf{U} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

Combining (4) and (7) with this last result we get,

$$(9) \qquad \lim_{n=\infty} \sum_a^b PQ = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

This proves the length AB in fig. 27 to be determinate whenever the area under the curve $y = \sqrt{1+[f'(x)]^2}$ is determinate: this area is determinate, by § 33, whenever the curve $y = \sqrt{1+[f'(x)]^2}$ is finite, continuous and single valued from a to b, inclusive. This curve is finite, continuous and single-valued, when $f'(x)$ is single-valued and continuous from a to b.

Hence, the length of the curve $y = f(x)$ *from* $x = a$ *to* $x = b$, *is determinate, according to our definition of length, when* $f'(x)$ *is single-valued and continuous from* a *to* b.

If we had treated y as independent variable throughout the foregoing investigation, and the equation of the curve had been expressed in the form $x = \phi(y)$, while the limiting values of y were a_1 and b_1, we should have obtained the result,

$$(10) \qquad \int_{a_1}^{b_1} \sqrt{1 + [\phi'(y)]^2}\, dy,$$

for the length AB.

57. DIFFERENTIAL OF LENGTH OF CURVE.

Let $y = f(x)$ be the equation of the curve APQ in fig. 29. Call the length of the curve, measured from the fixed point A to the variable point P, s. Then, clearly, s is some function of the abscissa $OM = x$. *We may find the differential of this unknown function of x (that is, the differential of s) as follows:* —

Give to x the increment $MN = \Delta x$; then s takes the corresponding increment, $\Delta s =$ arc PQ.

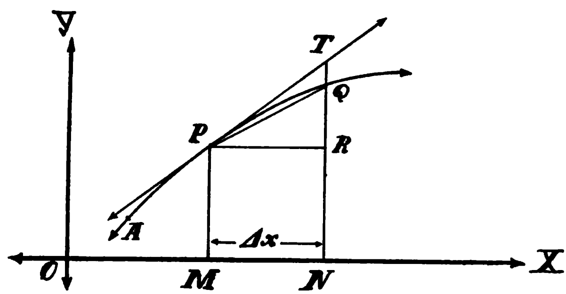

FIG. 29.

Let the curve be such that there is no cusp, nor point of inflexion, between P and Q; so that it lies wholly on one side of, and bends away from, its tangent PT. Then, assuming that the length of the arc PQ is greater than the length of its chord PQ; also, that the length of the arc PQ is less than the length of the broken line $PT + QT$; we have the following inequality:

$$(1) \qquad \sqrt{(\Delta x)^2 + (\Delta y)^2} < \Delta s < \sqrt{(\Delta x)^2 + (dy)^2} + dy - \Delta y;$$

since chord $PQ = \sqrt{PR^2 + RQ^2}$, $PT = \sqrt{PR^2 + RT^2}$, and $QT = RT - RQ = dy - \Delta y$, [see § 22].

Divide (1) by $\Delta x = dx$; and we get,

$$(2)\quad \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2} < \frac{\Delta s}{\Delta x} < \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \frac{dy}{dx} - \frac{\Delta y}{\Delta x}.$$

In (2) the ratio $\frac{dy}{dx} = \tan RPT$, is a constant as $\Delta x \doteq 0$; and $\lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \tan RPT = \frac{dy}{dx}$.

It follows, therefore, that $\lim_{\Delta x \doteq 0}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$;

and that, $\lim_{\Delta x \doteq 0}\left[\sqrt{1+\left(\frac{dy}{dx}\right)^2} + \frac{dy}{dx} - \frac{\Delta y}{\Delta x}\right] = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$;

and, from (2), we find,

$$(3)\quad \frac{ds}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta s}{\Delta x}\right] = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Equation (3) furnishes the *derivative of s with respect to x.* From it we obtain,

$$(4)\quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy;$$

which is the *differential of s.*

Equation (4) presents three different, equivalent, forms of the *differential of s.*

Using the second form we may express the two formulæ (9) and (10) § 56 as one, viz.:

$$(5)\qquad \int_a^b ds = \int_a^b \sqrt{dx^2+dy^2}.$$

In this form, either x or y may be the independent variable; but the limits, a and b, must be chosen to correspond with our choice of independent variable.

I. It is easy to prove from the foregoing that:—

The limit of the ratio of an infinitesimal arc to its chord, is unity.

For, from fig. 29, and from above, we have,

$$(6)\qquad \frac{arc\,PQ}{chord\,PQ} = \frac{\Delta s}{\sqrt{\Delta x^2+\Delta y^2}} = \frac{\frac{\Delta s}{\Delta x}}{\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}}.$$

From this it is evident that,

$$(7)\quad \lim_{\Delta x \doteq 0}\left[\frac{arc\ PQ}{chord\ PQ}\right] = \frac{\dfrac{ds}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = \frac{ds}{\sqrt{dx^2+dy^2}} = \frac{ds}{ds} = 1.$$

Exercises.

1. Find the length of the *straight line*, $y = mx + b$, from $x = 0$ to $x = a$. Obtain it by integration, and compare the result with the length found by a geometric method.

2. Find the length of a quadrant of the *circle*, $x^2 + y^2 = a^2$.

3. Find the perimeter of the *hypocycloid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $s = 6a$.

4. Find the length of the arc of the *parabola*, $x^2 = 2py$, from (0, 0) to the point (x_1, y_1).

Ans. $s = \dfrac{x_1}{2p}\sqrt{p^2 + x_1^2} + \dfrac{p}{2}\log\left[\dfrac{x_1 + \sqrt{p^2 + x_1^2}}{p}\right]$.

5. Adapt the result of ex. 4, by interchanging coördinates, so as to give the length of an arc of the *parabola*, $y^2 = 2px$, from (0, 0) to (x_1, y_1).

6. Find the *perimeter* of the segment cut off the *parabola*, $y^2 = 27x$, by the chord through (0, 0) and (3, 9).

Ans. 19.1152.

7. Find the arc of the *semi-cubical parabola*, $y^3 = a^2x^2$, from $(0, 0)$ to (x, y).

Ans. $s = \dfrac{8a^2}{27}\left[\left(1 + \dfrac{9y}{4a^2}\right)^{\frac{3}{2}} - 1\right]$.

8. Find the lengths of the arcs of the *cycloid*,

$$\left\{\begin{array}{l} x = a(\phi - \sin\phi) \\ y = a(1 - \cos\phi) \end{array}\right\},$$

from: (1) $\phi = 0$ to $\phi = \phi_1$, (2) $\phi = 0$ to $\phi = \pi$, and (3) $\phi = 0$ to $\phi = 2\pi$.

Ans. (1) $s = 4a\left(1 - \cos\dfrac{\phi}{2}\right)$, (2) $s = 4a$, (3) $s = 8a$.

9. Find the length of the arc of the *involute of the circle*,

$$\left\{\begin{array}{l} x = a(\cos\phi + \phi\sin\phi) \\ y = a(\sin\phi - \phi\cos\phi) \end{array}\right\},$$

from $\phi = 0$ to $\phi = \phi_1$.

Ans. $s = \frac{1}{2}a\phi_1^2$.

10. Find the length of the arc of the *catenary*, $y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$, from $x = 0$ to $x = x$.

$$Ans.\quad s = \frac{a}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right] = \sqrt{y^2 - a^2}.$$

11. Find the length of the arc of the *ellipse*,

$$\left\{\begin{array}{l} x = a\sin\phi \\ y = b\cos\phi \end{array}\right\},$$

from $\phi = 0$ to $\phi = \phi$.

$$Ans.\quad s = a\int_0^{\phi}\sqrt{1 - e^2\sin^2\phi}\,d\phi,\ where\ e^2 = \frac{a^2 - b^2}{a^2}.$$

This can be integrated by expanding $\sqrt{1 - e^2\sin^2\phi}$ into a power series in $e\sin\phi$, by the binomial series. The series will be convergent, since $e\sin\phi < 1$; and can be integrated term by term. The result will give s in the form of an infinite series.

If we integrate from $\phi = 0$ to $\phi = \frac{\pi}{2}$, we shall obtain a quadrant of the ellipse.

Multiplying by 4 will give the following series for the perimeter of an ellipse:

$$s = 2a\pi\left[1 - \left(\frac{1}{2}e\right)^2 - \frac{1}{3}\left(\frac{1\cdot 3}{2\cdot 4}e^2\right)^2 - \frac{1}{5}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^3\right)^2 - \ldots\right],$$

where $e = \frac{1}{a}\sqrt{a^2 - b^2}$.

An approximate formula for the perimeter of an ellipse, attributed to M. Barbarin, is,

$$s = \frac{4(a-b)^2 + 2\pi ab\sqrt{2}}{\sqrt{a^2 + b^2}}.$$

12. Compute the perimeter of the *ellipse*, $9x^2 + 25y^2 = 225$.

Ans. 25.6 . . . . .

13. Find the length of an arch of the *sinusoid*, $y = \sin x$.

$$\begin{aligned} Ans.\quad s &= 2\int_0^{\frac{\pi}{2}}[1 + \cos^2 x]^{\frac{1}{2}}\,dx \\ &= \pi\left[1 + \left(\frac{1}{2}\right)^2 - 3\left(\frac{1\cdot 1}{2\cdot 4}\right)^2 + 5\left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2 - \ldots\right] \\ &= 3.819423\ldots. \end{aligned}$$

[This series is alternating and converges slowly.]

*14. Find the formula for the length of an arc of the *epicycloid*,

$$\left\{ \begin{aligned} x &= (a+b)\cos\phi - b\cos\frac{a+b}{b}\phi \\ y &= (a+b)\sin\phi - b\sin\frac{a+b}{b}\phi \end{aligned} \right\}.$$

Ans. $s = \dfrac{4b(a+b)}{a}\left[\cos\dfrac{a}{2b}\phi_0 - \cos\dfrac{a}{2b}\phi_n\right].$

For a complete arch $\phi_0 = 0,\ \phi_n = \dfrac{2b}{a}\pi$; and $s = \dfrac{8b(a+b)}{a}$.

15. Find the length of an arch of the *hypocycloid*,

$$\left\{ \begin{aligned} x &= (a-b)\cos\phi + b\cos\frac{a-b}{b}\phi \\ y &= (a-b)\sin\phi - b\sin\frac{a-b}{b}\phi \end{aligned} \right\}.$$

Ans. $s = \dfrac{8b(a-b)}{a}.$

---

* In Exs. 14 and 15 a = radius of fixed circle, b = radius of rolling circle, and ϕ = angle between x-axis and the line joining O to the centre of the rolling circle.

CHAPTER XXIV.

AREAS OF SURFACES OF ROTATION.

58. DEFINITION OF AREA OF SURFACE OF ROTATION.

By *area of a rectangle* we mean the number obtained by multiplying the length of its base by the length of its altitude. The areas of all other rectilinear plane figures are measured by finding their equivalent rectangles. This is done by methods which depend in every case upon the possibility of proving the equivalence of surfaces by superposition. In defining and measuring the areas of plane surfaces which are bounded by curved lines, the method of limits is required. For example, the *area of a circle* is defined as the common limit of the areas of the inscribed and circumscribed regular polygons of n sides, when n is increased indefinitely.

The simplest curved surfaces are such as may be generated by rotating a plane curve about an axis in its plane; these are called **Surfaces of Rotation.**

Certain areas of surfaces of rotation (sphere, cone and cylinder) are defined in elementary geometry: and we shall use the formulæ for the lateral area of a conical frustum, as the basis of our treatment of surfaces of rotation. The formulæ are the following:

(1) $$S = \pi [R + R'] L,$$

(2) $$S = 2\pi R L.$$

In (1), R and R' are the radii of the bases, L is the slant height, and S is the lateral area of the frustum; while, in (2), R is the radius of the circular section parallel to the bases and midway between them, L is slant height, and S is lateral area.

Let us now define the area of the surface generated by rotating the arc AB (fig. 27, p. 137) about the x-axis.

Draw the series of ordinates, M_0A, M_1P_1, MP, etc., whose distances apart are all equal to $\Delta x = MN = \frac{M_0M_n}{n}$. Join the successive points AP_1, P_1P, PQ, etc., by chords; forming an unclosed inscribed polygon, $AP_1PQ \ldots . B$, of n sides.

Rotate this figure once about the x-axis. Each side of the polygon, as PQ, will generate the lateral surface of a frustum of a cone of rotation; whose area is given in (1) and (2) above. Hence, the perimeter of the n-sided polygon, $AP_1PQ \ldots . B$, will generate a surface, composed of n conical surfaces, which is defined for every integral value of n.

If the area of the surface generated by the perimeter of the inscribed polygon, $AP_1PQ \ldots . B$, approaches a limit when n is indefinitely increased, we shall call this limit the area of the surface generated by the arc AB.

We shall assume that this area (limit) exists when the perimeter of the inscribed polygon has a limit; and shall now proceed to calculate its differential element.

Represent by F (the initial of *Fläche*, German for surface) the area of a surface of rotation.

59. DIFFERENTIAL OF SURFACE OF ROTATION.

Let $y = f(x)$ be the equation of the generating curve; and suppose both $f(x)$ and $f'(x)$ are continuous and single-valued for the interval considered. Let APQ (fig. 29) be the curve. Take A a fixed point on it; and let P be a variable point whose abscissa is $OM = x$. The area, F, of the surface generated by the arc AP is, manifestly, some function of $OM = x$: *we wish to find its differential*, dF.

Give $OM = x$ the increment $MN = \Delta x$. The arc AP will take the corresponding increment, $\Delta s =$ arc PQ: and arc PQ will generate a zonal surface which will be the increment of F corresponding to Δx. Call the area of this zone ΔF. Then $RQ = \Delta y$, $RT = dy$, $NQ = y + \Delta y$, $NT = y + dy$, $PQ = \sqrt{\Delta x^2 + \Delta y^2}$, and $PT = \sqrt{\Delta x^2 + dy^2}$.

We shall assume that:

(1) $\Delta F >$ area of surface generated by chord PQ;
(2) $\Delta F <$ sum of areas generated by PT and QT. In brief,
(3) *area* $PQ < \Delta F <$ *area* $PT +$ *area* QT.

Now, $area\ PQ = \pi[MP + NQ]PQ$ [by § 58, (1)]

$$= \pi[2y + \Delta y]\sqrt{\Delta x^2 + \Delta y^2};$$

$$area\ PT = \pi[MP + NT]PT$$

$$= \pi[2y + dy]\sqrt{\Delta x^2 + dy^2};$$

and $area\ QT = \pi(NT)^2 - \pi(NQ)^2$

$$= \pi[(y + dy)^2 - (y + \Delta y)^2]$$

$$= \pi[2y + dy + \Delta y][dy - \Delta y].$$

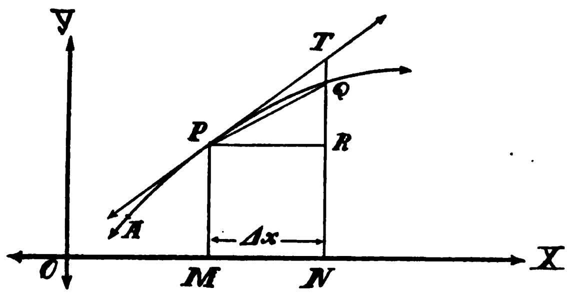

FIG. 29.

Substituting in (3) and dividing by $\Delta x = dx$, we get:

$$(4)\quad \pi[2y + \Delta y]\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} < \frac{\Delta F}{\Delta x} < \pi[2y + dy] \times$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} + \pi[2y + dy + \Delta y]\left(\frac{dy}{dx} - \frac{\Delta y}{\Delta x}\right).$$

Since $y = f(x)$ is continuous, both Δy and $dy \doteq 0$ when $\Delta x = dx \doteq 0$, and $\lim\limits_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx}$. Hence,

$$\lim_{\Delta x \doteq 0}\left[\pi(2y + \Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\right] = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

$$\lim_{\Delta x \doteq 0}\left[\pi(2y + dy)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

and $\lim\limits_{\Delta x \doteq 0}\left[\pi(2y + dy + \Delta y)\left(\frac{dy}{dx} - \frac{\Delta y}{\Delta x}\right)\right] = 0.$

It follows, therefore, that the first and last members of inequality (4) approach a common limit when $\Delta x \doteq 0$; and, hence, $\frac{\Delta F}{\Delta x}$, must approach the same limit. We get, therefore,

$$(5) \quad \frac{dF}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta F}{\Delta x}\right] = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

whence, by multiplying by dx. [See § 22.]

$$(6) \quad dF = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi y\ \sqrt{dx^2 + dy^2} = 2\pi y\,ds;$$

since $ds = \sqrt{dx^2 + dy^2}$, by § 57, (4).

This is the differential element of the area generated by rotating the given curve, $y = f(x)$, once about the x-axis.

If the rotation had taken place about the y-axis, we should have obtained in the same way the result,

$$(7) \qquad dF = 2\pi x\sqrt{dx^2 + dy^2} = 2\pi x\,ds,$$

for the differential element of the area generated by rotating a continuous curve once about the y-axis.

60. INTEGRAL FORMULÆ FOR AREAS OF SURFACES OF ROTATION.

The *differential element*, dF, of the area, F, generated by rotating the curve, $y = f(x)$, once about the x-axis, has been proved to be,

$$(1) \quad dF = 2\pi y\sqrt{dx^2 + dy^2} = 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

The problem of determining the area, F, generated by the arc between two fixed points, A and B, whose abscissæ are $x_0 = a$ and $x_n = b$, when dF is known, is identical with that of determining G, in § 33, when dG is known. We get, therefore,

$$(2) \quad F = \lim_{n=\infty} \sum_a^b 2\pi f(x)\ \sqrt{1 + [f'(x)]^2}\,dx$$

$$= 2\pi\int_a^b f(x)\ \sqrt{1 + [f'(x)]^2}\,dx;$$

from eqns. (8), (10) and (16) of § 33.

If the rotation takes place about the y-axis and the equation of the curve is in the form $x = f(y)$, the area generated by the arc from $y_0 = a'$ to $y_n = b'$, is,

(3) $$F = 2\pi \int_{a'}^{b'} f(y) \sqrt{1 + [f'(y)]^2}\, dy.$$

The most compact forms of F are,

(4) $$F = 2\pi \int_a^b y\, ds, \qquad \text{where} \quad ds = \sqrt{dx^2 + dy^2},$$

when the rotation is about the x-axis; and,

(5) $$F = 2\pi \int_a^b x\, ds,$$

when the rotation is about the y-axis.

In either of these forms we may choose either x or y for independent variable, as may be most convenient for anti-differentiation, taking care to choose the limits of integration to correspond with our choice of independent variable.

Formulæ (4) and (5) can be used perfectly well in the case where the equation of the curve is given in the form $x = f_1(\phi)$, $y = f_2(\phi)$; in which ϕ is the independent variable.

Example.

If the curve is $y = \sqrt{a^2 - x^2}$, we readily get,

$$dF = 2\pi \sqrt{a^2 - x^2}\, \frac{a}{\sqrt{a^2 - x^2}}\, dx = 2\pi a\, dx.$$

Whence,

$$F = 2\pi a \int_{-a}^{a} dx = 2\pi a x \Big|_{-a}^{a} = 2\pi a^2 - (-2\pi a^2) = 4\pi a^2;$$

which is the well known value of the area of the sphere whose radius is a.

Exercises.

1. Find the area of the *sphere* from the equations $\left\{\begin{matrix} x = a\cos\phi \\ y = a\sin\phi \end{matrix}\right\}$.

2. Find the area of the *circular spindle*, generated by revolving the circle $y = \sqrt{a^2 - x^2} - c$, $c < a$, about OX. Compute the area when $a = 8$ and $c = 4$.

$$\textit{Ans.} \quad (1) \quad F = 4\pi a\left[\sqrt{a^2 - c^2} - c\sin^{-1}\left(\frac{\sqrt{a^2 - c^2}}{a}\right)\right].$$
$$(2) \quad F = 275.456.$$

3. Find the area of the *ring surface* generated by revolving the circle, $y = b \pm \sqrt{a^2 - x^2}$, $a < b$, about OX.

Ans. $F = 4\pi^2 ab$.

4. Find the area of the surface generated by rotating the *parabola*, $y^2 = 2px$, about OX. Take the arc from $x = 0$ to $x = x$.

$$\textit{Ans.} \quad F = \frac{2\pi}{3}\left[(p + 2x)\sqrt{p^2 + 2px} - p^2\right].$$

5. The axis of a *parabolic reflector* is 20 inches long and the focus is $4\frac{1}{2}$ inches from the vertex. Find the area.

Ans. $F = 632\pi$.

6. Find the area generated by revolving the ellipse, $\left\{\begin{matrix} x = a\sin\phi \\ y = b\cos\phi \end{matrix}\right\}$ about (1) OX and (2) OY.

$$\textit{Ans.} \quad (1) \quad F = 2\pi b^2 + 2\pi ab\frac{\sin^{-1}e}{e},$$
$$(2) \quad F = 2\pi a^2 + \pi b^2 \frac{1}{e}\log\frac{1+e}{1-e}. \qquad e = \frac{1}{a}\sqrt{a^2 - b^2}.$$

7. What will it cost to cover an ellipsoidal foot-ball whose cross section through its axis, is the ellipse, $25x^2 + 36y^2 = 900$, at 2 cents per square inch?

Ans. \$7.14.

8. Find the area generated by the *four cusped hypocycloid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $F = \frac{12}{5}\pi a^2$.

9. Solve exercise 8, taking the equations, $\left\{\begin{matrix} x = a\cos^3\phi \\ y = a\sin^3\phi \end{matrix}\right\}$.

10. Find the area generated by the *catenary*, $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$, from $x = -1$ to $x = 1$.

Ans. $2\pi\,[e + 2 - e^{-1}] = 27.3354$.

11. Find the area generated by the cycloid, $\left\{\begin{matrix} x = a(\phi - \sin\phi) \\ y = a(1 - \cos\phi) \end{matrix}\right\}$.

Ans. $F = \frac{64}{3}\pi a^2$.

CHAPTER XXV.

POLAR COÖRDINATES.

AREAS AND LENGTHS OF PLANE CURVES: ANGLE BETWEEN RADIUS-VECTOR AND TANGENT: SUBTANGENT, SUBNORMAL, TANGENT AND NORMAL.

61. PLANE AREAS IN POLAR COÖRDINATES.

Let the curve AB in fig. 30 represent in polar coördinates the continuous function, $r = f(\theta)$. *It is required to define and calculate the area bounded by the arc P_0P_n and the radii-vectores OP_0 and OP_n.*

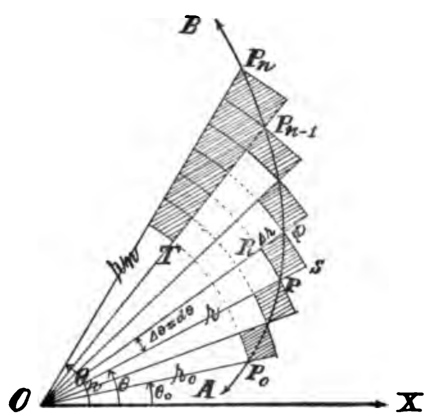

FIG. 30.

I. Divide $\angle P_0OP_n$ into n equal parts, $\frac{P_0OP_n}{n} = \Delta\theta$. Construct the two sets of circular sectors, as shown, each having its centre at O. Let **V** represent the sum of the areas of the overlapping sectors; and **U** the sum of the areas of the inscribed sectors. The difference,

$\mathbf{V}-\mathbf{U}$, equals the sum of the shaded curvilinear rectangles, of which $PRQS$ is one: and this sum is equal to the shaded portion, TP_n, of the sector $OP_{n-1}P_n$. That is, $\mathbf{V}-\mathbf{U}$ is less than the area of the sector $OP_{n-1}P_n$, which equals* $\frac{1}{2}r_n^2\Delta\theta$; since the radius is $OP_n = r_n$ and its arc is $r_n\Delta\theta$.

It is evident, also, that $\mathbf{V}$ is greater than $\mathbf{U}$; hence, we get,

$$(1) \qquad 0 < \mathbf{V}-\mathbf{U} < \tfrac{1}{2}r_n^2\Delta\theta.$$

Now increase n indefinitely. Inequality (1) will remain valid however large n may be made. But $\lim\limits_{n=\infty} \Delta\theta = \lim\limits_{n=\infty} \dfrac{P_0OP_n}{n} = 0.$

Hence, $\lim\limits_{n=\infty} \frac{1}{2}r_n^2\Delta\theta = 0$; and we get from (1),

$$(2) \qquad \lim_{n=\infty}[\mathbf{V}-\mathbf{U}] = 0; \text{ or } \lim_{n=\infty}\mathbf{V} = \lim_{n=\infty}\mathbf{U}.$$

The common limit of $\mathbf{V}$ *and* $\mathbf{U}$, *when* $n=\infty$, *is the area of the surface* OP_0P_n.

II. The *differential element of the area* OP_0P_n [fig. 30] may be found as follows: —

Let $\angle XOP = \theta$ be the vectorial angle of the point P. Give θ the increment $\Delta\theta = \angle POR$. The radius vector, $OP = r$, will take the corresponding increment $RQ = \Delta r$: and the area OP_0P will take the increment $OPQ = \Delta G$. Draw circular arcs, PR and SQ, with centres at O, and radii $OP = r$ and $OQ = r + \Delta r$. Then, the area of the circular sector $OPR = \frac{1}{2}r^2\Delta\theta$, and the area of $OSQ = \frac{1}{2}(r+\Delta r)^2\Delta\theta$.

Now, the figure shows that,

$$(3) \qquad \textit{area } OPR < \textit{area } OPQ < \textit{area } OSQ:$$

and hence,

$$(4) \qquad \tfrac{1}{2}r^2\Delta\theta < \Delta G < \tfrac{1}{2}(r+\Delta r)^2\Delta\theta.$$

Dividing (4) by $\Delta\theta$ we get,

$$(5) \qquad \frac{1}{2}r^2 < \frac{\Delta G}{\Delta\theta} < \frac{1}{2}(r+\Delta r)^2.$$

* It is proved in elementary geometry that: *The area of a circular sector is equal to half the product of its arc by its radius.*

Now $\lim\limits_{\Delta\theta \doteq 0} (r + \Delta r)^2 = r^2$; since $\lim\limits_{\Delta\theta \doteq 0} \Delta r = \lim\limits_{n = \infty} \Delta r = 0$: and we obtain from (5),

$$(6) \qquad \frac{dG}{d\theta} = \lim_{\Delta\theta \doteq 0} \left[\frac{\Delta G}{\Delta \theta}\right] = \frac{1}{2} r^2.$$

From (6) we get,

$$(7) \qquad dG = \frac{1}{2} r^2 d\theta ;$$

which is the differential element of area in polar coördinates.

III. We may now obtain the area OP_0P_n [fig. 30] by integration, as in § 33: hence,

$$(8) \quad G = \lim_{\Delta\theta \doteq 0} \sum_{\theta_0}^{\theta_n} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{\theta_0}^{\theta_n} r^2 d\theta = \frac{1}{2}\int_{\theta_0}^{\theta_n} [f(\theta)]^2 d\theta ;$$

which is the general integral formula for the area included by the polar curve, $r = f(\theta)$, and the radii-vectores corresponding to $\theta = \theta_0$ and $\theta = \theta_n$.

Example.

If the pole is on the circumference, and the polar axis passes through the centre, the polar equation of the circle whose radius is a, is $r = 2a\cos\theta$. One half the circle is traced while θ varies from 0 to $\frac{\pi}{2}$. We get, on substituting in (8),

$$(9) \ \frac{1}{2} \text{ area circle} = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4a^2 \cos^2\theta \, d\theta = 2a^2 \left[\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right]\Bigg|_0^{\frac{\pi}{2}}$$

$$= 2a^2\left[\frac{\pi}{4}\right] = \frac{\pi a^2}{2}.$$

Hence, the entire area is πa^2; which is the familiar formula for the area of a circle.

Exercises.

1. Find the area of each of the three loops of the curve, $r = 2a\sin 3\theta$.
   Ans. $G = \frac{1}{3}\pi a^2$.

2. Find the area of the curve, $r = a\sin 2\theta$.
   Ans. $G = \frac{1}{2}\pi a^2$.

3. Find the area of the *cardioid*, $r = a\cos^2\frac{\theta}{2}$.
   Ans. $G = \frac{3}{8}\pi a^2$.

4. Find the area of the *lemniscate*, $r^2 = a^2 \cos 2\theta$.

Ans. $G = a^2$.

5. Find the area of the *ellipse*, $r^2 = \dfrac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}$.

6. Find the whole area of the curve, $r = a(\sin 2\theta + \cos 2\theta)$.

Ans. $G = \pi a^2$.

7. Find the area of one circuit of the *spiral*, $r = a\theta$.

Ans. $G = \frac{4}{3}\pi^3 a^2$.

8. Find the area of one circuit of the *logarithmic* spiral, $r = e^{a\theta}$.

Ans. $G = \dfrac{1}{4a}\left[e^{4\pi a} - 1\right]$.

62. LENGTHS OF PLANE CURVES IN POLAR COÖRDINATES.

In polar coördinates, as in rectangular coördinates in § 55, we shall define "length" of a curve as: *The limit of the perimeter of an inscribed polygon when the number of sides becomes infinite and each side of the polygon is indefinitely diminished.*

The *differential element of length of curve* for polar coördinates may be obtained most simply by transformation of coördinates. The formulæ obtained in analytic geometry for transforming from rectangular to polar coördinates are,

(1) $$x = r\cos\theta, \qquad y = r\sin\theta.$$

In rectangular coördinates the differential, ds, of the length, s, of a curve is,

(2) $$ds = \sqrt{dx^2 + dy^2}. \qquad \text{[See § 57.]}$$

From (1) we get,

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta:$$

$$\therefore \quad dx^2 + dy^2 = dr^2 + r^2 d\theta^2 = ds^2.$$

Whence we get,

(3) $$ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

The length, s, of the curve $r = f(\theta)$ between the points corresponding to two given values of θ, say θ_0 and θ_n, can now be obtained from (3) by integration. We get, therefore,

(4) $$s=\int_{\theta_0}^{\theta_n}\sqrt{dr^2+r^2d\theta^2}=\int_{\theta_0}^{\theta_n}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

If r is taken as independent variable, (4) will be expressed in the form,

(5) $$s=\int_{r_0}^{r_n}\sqrt{1+r^2\left\{\frac{d\theta}{dr}\right\}^2}\,dr.$$

In some cases (5) is more readily evaluated than (4). This depends upon the form of the function, $r=f(\theta)$.

Exercises.

1. Find the circumference of the *circle*, $r = 2a\cos\theta$.

Ans. $s = 2\pi a$.

2. Find the perimeter of the *cardioid*, $r = a\cos^2\frac{\theta}{2}$.

Ans. $s = 4a$.

3. Find the length of one circuit of the *logarithmic spiral*, $r = e^{a\theta}$.

Ans. $s = \frac{e^{2\pi a}-1}{a}\sqrt{a^2+1}$.

4. Find the length of one circuit of the *spiral of Archimedes*, $r = a\theta$.

Ans. $s = \pi a\sqrt{1+4\pi^2}+\frac{a}{2}\log\left[2\pi+\sqrt{1+4\pi^2}\right]$.

5. Find the formula for the length of an arc of the *hyperbolic spiral*, $r=\frac{a}{\theta}$.

Ans. $s = \left\{a\log\left[\theta+\sqrt{1+\theta^2}\right]-\frac{a\sqrt{1+\theta^2}}{\theta}\right\}\Bigg|_{\theta_1}^{\theta_2}$.

62a. ANGLE BETWEEN RADIUS-VECTOR AND TANGENT, AT A GIVEN POINT ON A POLAR CURVE.

Let the equation of the curve be $r = f(\theta)$. Take any point $P(r, \theta)$ on it. The radius-vector of P will make the angle θ with the polar axis: and the tangent at P will make an angle with the polar axis which we may call τ. [See fig. 31.]

Call the angle between the radius-vector and the tangent, α. From the figure we may see that,

$$(1) \qquad \alpha = \tau - \theta.$$

Whence, by trigonometry,

$$(2) \qquad \tan\alpha = \frac{\tan\tau - \tan\theta}{1 + \tan\tau \tan\theta}.$$

A differential formula for tan α may be obtained as follows:—

Taking the pole as origin and the polar axis for x-axis, we shall have, at the point P,

$$(3) \qquad y = r\sin\theta, \ x = r\cos\theta.$$

But, from § 11, we have,

$$(4) \qquad \tan\tau = D_x y = \frac{dy}{dx}, \ [\S\ 22].$$

From (3) we get
$$dy = \sin\theta dr + r\cos\theta\, d\theta,$$
$$dx = \cos\theta dr - r\sin\theta d\theta:$$

$$(5) \qquad \therefore \quad \tan\tau = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta}.$$

Putting this value of tan τ in (2), and reducing, we get,

$$(6) \qquad \tan\alpha = \frac{rd\theta}{dr} = r\,\frac{d\theta}{dr}.$$

Formula (5) *is a general formula for the slope of the tangent to the curve,* $r = f(\theta)$, *at any point* (r, θ) *on the curve.*

Formula (6) *is a general formula for the tangent of the angle,* α, *which the radius-vector of the point* (r, θ) *makes with the tangent to the curve at that point.*

62b. POLAR SUBTANGENT, SUBNORMAL, TANGENT AND NORMAL.

In fig. 31, let $P(r, \theta)$ be a point on the curve AB, whose equation is $r = f(\theta)$. Let PN be the normal, and TPt be the tangent at P. Draw TON perpendicular to OP at O, meeting the tangent and normal at T and N. Draw OQ perpendicular to the tangent PT.

Special names are given to the lines PT, PN, OT, and ON, as follows:

PT is called the	**polar tangent**	at P;
PN	"	**polar normal**	" P;
OT	"	**polar subtangent**	" P;
ON	"	**polar subnormal**	" P.

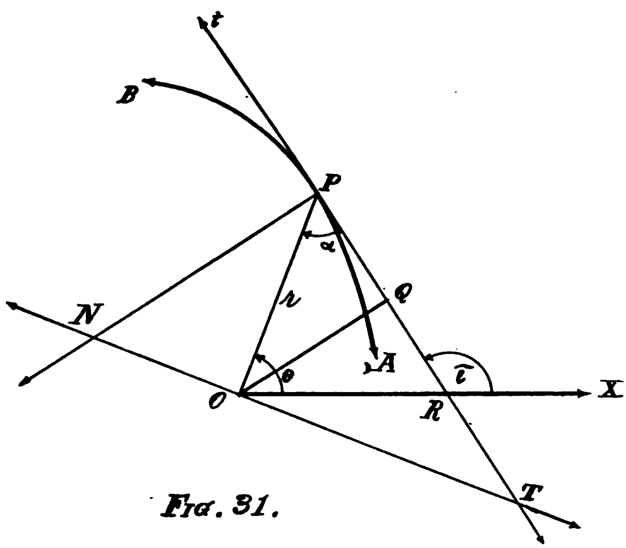

Fig. 31.

Differential expressions may be obtained for these lines as follows:

(1) $OT = OP\tan\alpha = r\tan\alpha = r^2 \dfrac{d\theta}{dr}$, [§ 62 a, (6)].

(2) $PT = OP\sec\alpha = r\sqrt{1 + \tan^2\alpha} = r\sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}$

$$= \frac{r\sqrt{dr^2 + r^2 d\theta^2}}{dr} = r\frac{ds}{dr};$$

since $\sqrt{dr^2 + r^2 d\theta^2} = ds$, by (3), § 62.

(3) $ON = OP \tan OPN = r \tan(90 - \alpha) = r \operatorname{ctn} \alpha$

$$= \frac{r}{\tan \alpha} = \frac{r}{r\dfrac{d\theta}{dr}} = \frac{dr}{d\theta}.$$

(4) $PN = \sqrt{OP^2 + ON^2} = \sqrt{r^2 + \left\{\dfrac{dr}{d\theta}\right\}^2}$

$$= \frac{\sqrt{r^2 d\theta^2 + dr^2}}{d\theta} = \frac{ds}{d\theta}.$$

Hence, we have,

(5) $PT = \textit{polar tangent} = r\dfrac{ds}{dr}$;

(6) $PN = \textit{polar normal} = \dfrac{ds}{d\theta}$;

(7) $OT = \textit{polar subtangent} = r^2\dfrac{d\theta}{dr}$;

(8) $ON = \textit{polar subnormal} = \dfrac{dr}{d\theta}$.

Representing the perpendicular, OQ, from the pole to the tangent by p, we have,

(9) $p = OQ = OP \sin \alpha = r\dfrac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}} = r\dfrac{r\dfrac{d\theta}{dr}}{\sqrt{1 + r^2\left\{\dfrac{d\theta}{dr}\right\}^2}}$

$$= \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}} = r^2\frac{d\theta}{ds}.$$

Exercises.

1. In the curve, $r = 2a \sin \theta$, show that $\alpha = \theta$. Plot the curve. What curve is it?

2. In the *lemniscate*, $r^2 = a^2 \cos 2\theta$, show that: $\alpha = \frac{\pi}{2} + 2\theta$; $PT = r \csc 2\theta$; $PN = \frac{a^2}{r}$; $OT = -a \operatorname{ctn} 2\theta \sqrt{\cos 2\theta}$; and $ON = -\frac{a^2}{r}\sin 2\theta$.

3. The general polar equation of a *conic section* is $\frac{l}{r} = 1 + e\cos\theta$, the pole being at a focus, and e being eccentricity.

Show that: $\tan\alpha = \frac{l}{er\sin\theta}$; $\quad PT = \frac{r\sqrt{1+2e\cos\theta+e^2}}{e\sin\theta}$;

$$PN = \frac{r^2}{l}\sqrt{1+2e\cos\theta+e^2}; \quad OT = \frac{l}{e\sin\theta}; \text{ and } ON = \frac{er^2\sin\theta}{l}.$$

If $e < 1$, the conic section is an ellipse; if $e = 1$, it is a parabola; and if $e > 1$, it is an hyperbola.

4. Show, for the parabola, that: the equation of the curve is $r = \frac{l}{2}\sec^2\frac{\theta}{2}$; $\tan\alpha = \operatorname{ctn}\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$; $PT = r\operatorname{cosec}\frac{\theta}{2}$; $PN = r\sec\frac{\theta}{2}$; $OT = l\operatorname{cosec}\theta$; and $ON = r\tan\frac{\theta}{2}$.

These results may be derived from the results of Ex. 3.

5. The polar equation of an ellipse, pole at center, is $r^2(1 - e^2\cos^2\theta) = b^2$. Show that: $ON = \mp be^2\sin\theta\cos\theta(1 - e^2\cos^2\theta)^{-\frac{1}{2}}$;

$$OT = \frac{b(1 - e^2\cos^2\theta)^{\frac{1}{2}}}{e^2\sin\theta\cos\theta}; \text{ and } \tan\alpha = \operatorname{ctn}\theta - \frac{1}{e^2}\sec\theta\csc\theta.$$

CHAPTER XXVI.

CURVATURE OF PLANE CURVES.

DEFINITIONS: RECTANGULAR AND POLAR FORMULÆ FOR CURVATURE: SECOND DIFFERENTIALS.

63. DEFINITION OF CURVATURE FOR PLANE CURVES.

Consider the arc AB of any continuous plane curve, which has neither cusp nor point of inflexion between A and B. [See fig. 32.] The arc may be conceived as generated by the point P, moving from A to B: and *the direction of the motion of P at any instant, is the direction of the tangent at P, and may be defined as the direction of the curve at P.*

The curve possesses a quality called *bend*, or *flexure*, or *turn*, which is determined by the change of direction of the curve from point to point: and the bend of an arc of given length is greater or less, according as the angle between the tangents at its ends is greater or less; provided the turn is, throughout, toward one side of the arc.

The bend, or flexure, of an arc of a circle may be called uniform, or constant; since the tangents at the ends of any two arcs of equal lengths on equal circles, make equal angles with each other. But, *in an ellipse*, if two arcs of equal lengths be taken, it is plain that the tangents drawn at the ends of each will not, in general, make equal angles with each other. In this case, *and in general, the flexure is variable.*

We wish to devise a method of measuring the flexure of a given curve at a given point: and we shall call the result of the measurement, or, *the number which expresses the measure of flexure*, **the curvature at the given point.**

We have to show that flexure, or bend, may be measured.

Let us examine, first, the circle of radius a. It is evident that, whatever be the method devised for measuring the flexure of the circle, the following conditions must be satisfied by the result:—

(1) if a increases the curvature decreases, and conversely; (2) if a is fixed, curvature is fixed, and conversely.

A result satisfying both these conditions may be obtained as follows: — Take an arc, AB, of the circle; and call its length, s. Draw its tangents at A and B; and call the angle between them, α. If O is the centre, and radii OA and OB are drawn to its ends, it is easy to prove that $\angle AOB = \alpha$. But, the length of an arc of a circle equals the product of its radius by the circular measure of its subtended angle; hence, $s = a \times AOB$; or,

(1) $$s = a\alpha.$$

Now divide the angle α (expressed in circular measure) by the length of the arc, s. We get from (1),

(2) $$\frac{\alpha}{s} = \frac{\alpha}{a\alpha} = \frac{1}{a}.$$

If s is decreased indefinitely, α will also decrease indefinitely, since $\alpha = \frac{s}{a}$; hence, s and α will vanish together. But the ratio $\frac{\alpha}{s} = \frac{1}{a}$ is constant if a is fixed; moreover, this ratio will decrease as a increases, and conversely. Hence we may take its value, $\frac{1}{a}$, as the measure of the flexure of the circle whose radius is a. *If, therefore, we represent by k the curvature of the circle of radius a, we have,*

(3) $$k = \frac{1}{a}.$$

In an analogous way we shall define curvature for any given plane curve as follows: —

I. *Let P and Q be the ends of an arc, PQ; which, from P to Q, bends away from the tangent at P. Call its length, s; and let α represent the circular measure of the angle between its tangents at P and Q (this is sometimes called *angle of contingence of the arc). The ratio, $\frac{\alpha}{s}$, is called* **the mean curvature** *of the arc PQ.*

If we denote *mean curvature by k_m*, we have, by definition,

(4) $$k_m = \frac{\alpha}{s}.$$

* The angle of contingence is equal to the angle between the normals at the ends of the arc.

II. *In the preceding definition, conceive the length, s, of the arc, PQ, to diminish indefinitely by causing Q to approach the fixed point, P. If the mean curvature, $k_m = \frac{a}{s}$, approaches a limit when s approaches zero (that is, when Q approaches P), this limit will be called* **the curvature at the point P.**

If we denote *the curvature at P* by k, we have, by definition,

$$(5) \qquad k = \lim k_m = \lim_{s \doteq 0} \left[\frac{a}{s}\right].$$

We have developed a method of measuring a in § 11; and, in § 56, a general method of calculating s: hence, the value of mean curvature, k_m, can, in general, be found when the equation of the curve is known.

We shall establish general formulæ in the next section by which k may be obtained.

64. GENERAL FORMULÆ FOR THE CURVATURE OF A PLANE CURVE.

I. Let the curve AB in fig. 32 be the locus of the function $y = f(x)$; which has a continuous and single-valued derivative, $f'(x)$, for all values of x belonging to the interval (a_1, b_1). Take $P(x, y)$ any point whose abscissa belongs to the interval (a_1, b_1); and let $\tau = \angle XRP$ be the angle between OX and the tangent at P.

It has been shown in § 11 that,

$$(1) \qquad \tan\tau = f'(x); \text{ or, } \tau = \tan^{-1} f'(x).$$

If $f'(x) = m$ is a constant in the interval (a_1, b_1), the function $y = f(x)$ is linear [§ 45, I]; and AB is a straight line: conversely, if the function is linear, its locus is a straight line; and $f'(x) = m$ is a constant. In this case, $\tau = \tan^{-1} m$ is a constant: but, *in general, $f'(x)$ and, therefore, τ are functions of x.*

Give to x an increment, Δx; and let $\Delta s = \text{arc} PQ$, and $\Delta\tau = tQ't' = XR'Q' - XRQ'$, be the corresponding increments of $s = \text{arc} AP$, and of $\tau = XRP$.

The mean curvature of the arc PQ is,

$$(2) \qquad \frac{\angle tQ't'}{\text{arc} PQ} = \frac{\Delta\tau}{\Delta s} = \frac{\Delta\tau}{\Delta x} \div \frac{\Delta s}{\Delta x}, \text{ by I, §63.}$$

The curve being continuous, $\Delta s \doteq 0$ when $\Delta x \doteq 0$; and, from (2), we obtain the following result for the curvature at P [by II, § 63]:

$$(3)\ k = \lim_{\Delta s \doteq 0}\left[\frac{\Delta \tau}{\Delta s}\right] = \lim_{\Delta x \doteq 0}\left[\frac{\Delta \tau}{\Delta s}\right] = \lim_{\Delta x \doteq 0}\left[\frac{\Delta \tau}{\Delta x}\right] \div \lim_{\Delta x \doteq 0}\left[\frac{\Delta s}{\Delta x}\right].$$

$$(4)\ \therefore \qquad k = \frac{d\tau}{ds} = \frac{d\tau}{dx} \div \frac{ds}{dx}.$$

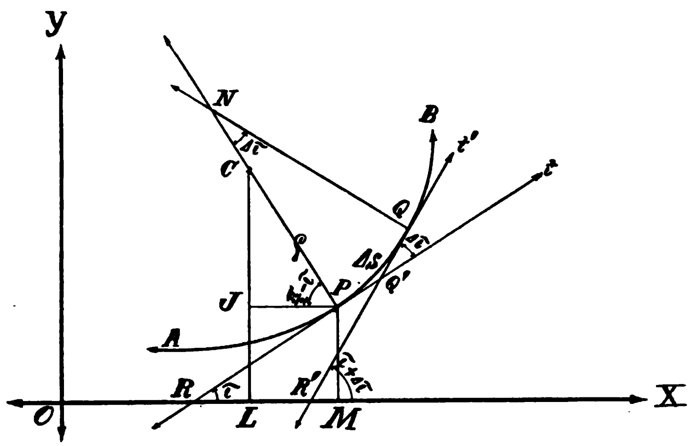

FIG. 32.

II. The expression for $\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}$ has been established in § 57. It can always be found when $y = f(x)$ is given, if the derivative, $f'(x)$, can be found.

We will now show that $\frac{d\tau}{dx}$ can be found, in general, under the same conditions. We have noted above that τ is, in general, a function of x. Differentiating the expression $\tau = \tan^{-1} f'(x)$ by XXIII, § 23, we get,

$$(5)\quad d\tau = \frac{d[f'(x)]}{1 + [f'(x)]^2}\,; \text{ or, } \frac{d\tau}{dx} = \frac{f''(x)}{1 + [f'(x)]^2}\,.$$

The second form of (5) is obtained by dividing the first by dx, then putting $f''(x)$ for $\frac{d[f'(x)]}{dx}$; since each of the latter expressions denotes the first derivative of $f'(x)$, or the second derivative of $f(x)$.

Substituting in (4) the values found for $\frac{ds}{dx}$ and $\frac{d\tau}{dx}$, we obtain *the following general formula for the curvature of the locus of* $y = f(x)$ *at the point* (x, y):

$$(6) \qquad k = \frac{f''(x)}{[1 + \{f'(x)\}^2]^{\frac{3}{2}}} = \frac{D_x^2 y}{[1 + (D_x y)^2]^{\frac{3}{2}}}.$$

The second form of (6) is obtained by noting that $D_x y \equiv f'(x)$, and that $D_x^2 y \equiv f''(x)$.

As examples illustrating the application of (6), let us find k for the line, $y = mx + b$; and for the circle, $x^2 + y^2 = a^2$.

For $y = mx + b$, $D_x y = m$, and $D_x^2 y = 0$; hence, $k = \frac{0}{[1 + m^2]^{\frac{3}{2}}} = 0$; and *the curvature of a straight line is zero.*

Prove the converse.

For $x^2 + y^2 = a^2$, $y = \pm\sqrt{a^2 - x^2}$. The upper sign refers to the semi-circumference *above* the x-axis, and the minus sign to that *below* the x-axis. These may be treated separately.

Take $y = \sqrt{a^2 - x^2}$; then $D_x y = \frac{-x}{\sqrt{a^2 - x^2}}$, and $D_x^2 y = \frac{-a^2}{[a^2 - x^2]^{\frac{3}{2}}}$.

Whence, $k = \frac{-1}{a}$; *and the curvature of the upper semi-circumference, which is concave downwards, is the negative of the reciprocal of the radius.*

Taking $y = -\sqrt{a^2 - x^2}$, we get $D_x y = \frac{x}{\sqrt{a^2 - x^2}}$, $D_x^2 y = \frac{a^2}{[a^2 - x^2]^{\frac{3}{2}}}$, and $k = \frac{1}{a}$.

This semi-circumference is *convex downwards.* Its curvature is numerically equal to that of the upper half of the circumference: and the results agree with those obtained for the circle in § 63.

III. We should have anticipated a difference in the sign of k between a concave and convex curve, from the fact that $D_x^2 y$ is negative when the curve is concave downwards, and positive when it is concave upwards. [See Ex. 1, § 21.]

Hence, in general, the curvature, k, as obtained by formula (6), will be positive when the curve is convex downwards and negative when it is concave downwards. In the first case the curve lies above its tangents; and in the second case, below.

It may be shown from this property of k that $k = 0$, and therefore $D_x^2 y = 0$, at a point of inflexion. [Compare § 21.]

[*Work Exs. 1, 2, and 3, on* p. 169.]

IV. Formula (6) can be conveniently used in all cases when the equation of the curve is given in the form of a single equation connecting x and y, which can be solved for y so as to take the form $y = f(x)$. It is not so convenient for use in cases when the equation of the curve is given in the form, $x = f_1(\phi)$, $y = f_2(\phi)$; where x and y are expressed in terms of a third variable. For this case, the differential notation gives a better form of expression for k; for reasons to be seen below.

V. We have seen in § 22 that $D_x y = \dfrac{dy}{dx}$: we must now express $D_x^2 y$ in terms of the differential notation, *where we are to understand by $D_x^2 y$, the ratio of the differential of $D_x y$ to the differential of x.*

Let us find, first, the differential of $D_x y = \dfrac{dy}{dx}$; in which we must regard $\dfrac{dy}{dx}$ as a quotient, and use the formula of Ex. 8c, § 9. We get,

$$*(7) \qquad d\,[D_x y] \equiv d\left[\frac{dy}{dx}\right] = \frac{dx\,d^2y - dy\,d^2x}{dx^2};$$

where $d^2y \equiv d\,[dy]$ and $d^2x \equiv d\,[dx]$ denote *the differentials of the differentials of y and x.* These are called **second differentials.**

Dividing (7) by dx, we obtain,

$$(8) \qquad D_x^2 y = \frac{d\,[D_x y]}{dx} = \frac{d\left[\dfrac{dy}{dx}\right]}{dx} = \frac{dx\,d^2y - dy\,d^2x}{dx^3}.$$

Substituting in (6) we have,

$$(9) \qquad k = \frac{\dfrac{dx\,d^2y - dy\,d^2x}{dx^3}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

Form (9) is the most general differential formula for k, and is applicable in the case when x and y are given as functions of a third variable.

VI. For example, the equation of the ellipse is sometimes given in the form, $x = a\cos\phi$, $y = b\sin\phi$; where ϕ is the eccentric angle of the point (x, y); and ϕ is regarded as independent variable.

---

* The student should note that dx^2 means $(dx)^2$, *not* $d(x^2)$.

To find k for the ellipse, when its equation is in this form, we must first show that: —

The second differential of the independent variable is zero.

In this example ϕ is independent; hence we are to show that $d^2\phi = d[d\phi] = 0$. But the principle is a general one; and applies to all cases, whatever be the symbol representing the independent variable.

By the definition of differential in § 22, we must have,

$$(10) \qquad d^2\phi \equiv d[d\phi] = D_\phi(d\phi)\Delta\phi;$$

where $d\phi$ and $\Delta\phi$ are the increments of ϕ in the two successive, independent, operations of calculating the first and second derivatives of the given function of ϕ.

Now, $d\phi$, being arbitrary, may be so chosen that *it shall not be a function of ϕ; and, therefore, so that its increment, $\Delta(d\phi)$, shall be zero in the second operation above-mentioned.*

Hence $D_\phi(d\phi) = \lim_{\Delta\phi \doteq 0}\left[\frac{\Delta(d\phi)}{\Delta\phi}\right] = 0$; since $\Delta(d\phi) = 0$; and, therefore $d^2\phi = 0$, from (10), as was to be proved.

It may be noted that the increment, $\Delta\phi$, in the second operation, is arbitrary. *It may, therefore, be taken equal to the increment, $d\phi$, of the first operation. It is so regarded when ϕ is independent.*

Returning to the equations $x = a\cos\phi$, $y = b\sin\phi$, we obtain,

$$dx = -a\sin\phi\, d\phi, \quad dy = b\cos\phi\, d\phi,$$
$$d^2x = -a\cos\phi\, d\phi^2, \quad d^2y = -b\sin\phi\, d\phi^2.$$

Substituting in (9) we get, after reduction,

$$(11) \qquad k = \frac{-ab\operatorname{cosec}^3\phi}{(a^2 + b^2\operatorname{ctn}^2\phi)^{\frac{3}{2}}}.$$

This gives the curvature at any point, P, of the ellipse, in terms of its eccentric angle, ϕ.

[*Work Exs. 4 and 5 on* p. 170.]

VII. The formula for curvature in *polar coördinates* may be obtained from (9) by the transformation, $x = r\cos\theta$, $y = r\sin\theta$; where (r, θ) are the polar coördinates of the point (x, y).

Regarding θ as independent, we obtain,

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta, \quad dy = \sin\theta\, dr + r\cos\theta\, d\theta,$$
$$d^2y = \sin\theta\, d^2r + 2\cos\theta\, d\theta\, dr - r\sin\theta\, d\theta^2,$$
$$d^2x = \cos\theta\, d^2r - 2\sin\theta\, d\theta\, dr - r\cos\theta\, d\theta^2.$$
$$\therefore\ dx^2 + dy^2 = dr^2 + r^2 d\theta^2,$$
$$dx d^2y - dy d^2x = (r^2 d\theta^2 + 2dr^2 - r d^2r)\, d\theta.$$

Substituting in (9) and reducing we get,

$$(12) \qquad k = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\,\dfrac{d^2r}{d\theta^2}}{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Form (12) gives the general expression for curvature in polar coördinates; and may be used in all cases when the equation of the curve is given in the form $r = f(\theta)$.

VIII. Formula (8) assumes a simpler form in all cases when x is independent; for, as shown above, in such cases $d^2x = d(dx) = 0$; and (8) reduces to,

$$(13) \qquad D_x^2 y = \frac{d^2y}{dx^2}\,; \quad \text{or,} \ d^2y = D_x^2 y\, dx^2.$$

Equation (13) may be regarded as defining the symbol $\dfrac{d^2y}{dx^2}$ *to be the second derivative of* $y = f(x)$ *with respect to* x, *when* x *is independent variable;* but, when x is *not* independent, form (8) defines the differential form of the second derivative of y with respect to x.

The second form of (13) defines the symbol, d^2y, or the second differential of y, for both cases; the value of $D_x^2 y$ is to be taken from (8) when x *is not* independent; and, from (13) when x *is* independent.

IX. Making the assumption that x is independent variable, gives the two following forms for k, on putting $d^2x = 0$ in (9):

$$(14) \qquad k = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{dx d^2y}{(dx^2 + dy^2)^{\frac{3}{2}}}.$$

These forms are equivalent to those in (6) *when x is independent.*

X. Two special cases are worth noting.

a) *If the curve* $y = f(x)$ *is parallel to the x-axis at P*, then $\frac{dy}{dx} = 0$; and we get from (14) the simpler form,

$$(15) \qquad k = \frac{d^2y}{dx^2},$$

for the curvature at P.

b) If the curve $y = f(x)$ is a very flat one, nearly parallel to the x-axis, then $\frac{dy}{dx} = m$ is a very small quantity; its square is smaller still; and if we develop $[1 + m^2]^{3/2}$ by the binomial theorem we get,

$[1 + m^2]^{\frac{3}{2}} = 1 + \frac{3}{2} m^2 + \frac{1 \cdot 3}{2 \cdot 4} m^4 +$ terms containing increasingly higher powers of m^2.

It is evident, therefore, if $m = \frac{dy}{dx}$ is very small, that *unity* is a fairly good approximation to the value of $[1 + m^2]^{\frac{3}{2}}$. *In such case, form* (15) *furnishes a close approximation to the curvature, k.*

This approximate value of k, viz. $\frac{d^2y}{dx^2}$, is used in many problems concerning the curvature of columns and beams; where the column or beam bends under its load; and where, from the nature of the case, only a slight degree of flexure is consistent with the safety of the structure resting on the column or beam.

XI. We have seen in II above that the curvature of the circle is a constant, not zero*: it may be shown, conversely, that if the curvature of a curve is a constant, not zero, the curve must be a circle; and, hence, *the curve of constant curvature must be a circle.*

This may be proved as follows: —

Let $\frac{1}{k} = a =$ a constant. Then $k = \frac{1}{a} = \frac{d\tau}{ds}$ [see equation (4) of this section]. Hence, $ds = a\,d\tau$.

---

* The case when the curvature is zero has been shown in the example under II to belong to the straight line.

But, since $\tan\tau = \dfrac{dy}{dx}$, we get,

$$(16)\quad \sin\tau = \frac{\tan\tau}{\sqrt{1+\tan^2\tau}} = \frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = \frac{dy}{\sqrt{dx^2+dy^2}} = \frac{dy}{ds};$$

$$(17)\quad \cos\tau = \frac{1}{\sqrt{1+\tan^2\tau}} = \frac{dx}{\sqrt{dx^2+dy^2}} = \frac{dx}{ds}.$$

From (17) and (16) we have,

$$dx = \cos\tau\, ds = a\cos\tau\, d\tau,$$
$$dy = \sin\tau\, ds = a\sin\tau\, d\tau.$$

From these equations we get, by anti-differentiation, the following: $x = a\sin\tau + b$, $y = -a\cos\tau + c$; where b and c are arbitrary constants [§ 45, III]. From these equations we get,

$$\sin\tau = \frac{x-b}{a}, \quad \cos\tau = \frac{y-c}{-a}.$$

But $\sin^2\tau + \cos^2\tau = 1$, from trigonometry; hence,

$$(x-b)^2 + (y-c)^2 = a^2,$$

which is the equation of the locus of (x, y) when the curvature is constant. This is the circle whose centre is (b, c) and radius is a.

Exercises.

Find the curvature of the following curves at the point $P(x, y)$: —

1. The *parabola*, $y^2 = 4px$.

Ans. $k = \dfrac{\mp\sqrt{p}}{2(p+x)^{\frac{3}{2}}}$.

2. The *equilateral hyperbola*, $xy = a$.

Ans. $k = \dfrac{2ax^3}{(a^2+x^4)^{\frac{3}{2}}}$.

3. The *catenary*, $y = \dfrac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$.

Ans. $k = \dfrac{4}{a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2}$.

4. The *hyperbola*, $x = a \sec \phi$, $y = b \tan \phi$.

Ans. $k = \dfrac{-ab\cos^3\phi}{(a^2 \sin^2\phi + b^2)^{\frac{3}{2}}}$.

5. The *astroid, or four-cusped hypocycloid*, $x = a\cos^3\phi$, $y = a\sin^3\phi$.

Ans. $k = \dfrac{1}{3a} \sec\phi \operatorname{cosec}\phi$.

[*Suggestion: — Use form (9).*]

6. The *spiral of Archimedes*, $r = a\theta$.

Ans. $k = \dfrac{2 + \theta^2}{a(1 + \theta^2)^{\frac{3}{2}}}$.

7. The *logarithmic spiral*, $r = e^{a\theta}$.

Ans. $k = \dfrac{1}{r\sqrt{1 + a^2}}$.

8. The *lemniscate*, $r^2 = a^2 \cos 2\theta$.

Ans. $k = \dfrac{3r}{a^2}$.

CHAPTER XXVII.

CIRCLE OF CURVATURE, RADIUS AND CENTRE OF CURVATURE.

65. CIRCLE, RADIUS, AND CENTRE OF CURVATURE AT A GIVEN POINT OF A PLANE CURVE.

I. Let $y = f(x)$ be a continuous, single-valued function of x throughout an interval (a, b); then, in general, both $f'(x)$ and $f''(x)$ will assume determinate values, $f'(x_1)$ and $f''(x_1)$, for a given value, x_1, in that interval. Hence, the value of curvature, k, from equation (6) § 64, is determinate, in general, at the point on the locus of $y = f(x)$ corresponding to $x = x_1$.

We have shown in § 64 that the curvature of a circle is the reciprocal of its radius; hence, its radius is the reciprocal of its curvature. Now, a circle may be drawn with any given radius; we may, therefore, draw a circle having any given curvature.

Turn, now, to fig. 32, p. 173, in which P is to be considered a fixed point on the locus, AB, of the function $y = f(x)$. Let Pt and PN be the tangent and normal at P. Let k be the curvature at P. On the normal PN, on the side of the curve opposite to the tangent at P, lay off a length $PC = \frac{1}{k}\cdot$ With the centre C, and radius PC, conceive a circle to be drawn. This circle will touch the line Pt at P, and will, therefore, be tangent to the curve AB at P: and since its radius is $PC = \frac{1}{k}$, *it will have the same curvature as the arc AB at P.*

This particular circle, tangent to the curve AB at P, having its centre on the inner portion of the normal at P, and its curvature equal to the curvature of AB at P, is called **the circle of curvature at P**; *its centre is called* **the centre of curvature at P**; *and its radius is* **the radius of curvature at P.** *It is called, also,* **the osculating circle at P.**

II. *Denote the radius of curvature by ρ.* Then we get, from § 64, (9), the following general rectangular formula for ρ:

$$(1) \qquad \rho = \frac{1}{k} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{dx d^2y - dy d^2x}{dx^3}}.$$

This formula is applicable both to the case when the equation of the curve is given in the form, $x = f_1(\phi)$, $y = f_2(\phi)$, and ϕ is the independent variable, as well as to the case when the equation is given in the form $y = f(x)$, in which x is independent variable. But when x is independent $d^2x = 0$; and (1) takes the following form:

$$(2) \qquad \rho = \frac{1}{k} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx d^2y}.$$

It may be seen that ρ gives a positive result (since $\frac{d^2y}{dx^2}$ is positive) at the point P whenever the curve, on both sides of P, lies *above* the tangent at P; and that ρ gives a negative result when the curve lies *below* its tangent (since, in the last case, $\frac{d^2y}{dx^2}$ is negative).

At a point of inflexion, $k = 0$ [§ 64, III]; *hence* $\rho = \pm \infty$ *at a point of inflexion.*

Also, $k = 0$, and $\rho = \pm \infty$, if $y = f(x)$ is a linear equation [§ 64, II]; *hence the radius of curvature of a straight line is infinite.*

It follows from § 64, XI, *that the only curve whose radius of curvature, ρ, is a constant, $a \gtrless 0$, is the circle whose radius is a.*

If $\rho = a = 0$, then $k = \frac{1}{\rho} = \infty$; that is, *the circle of zero radius (or a point) has an infinite curvature, with a zero radius of curvature: and conversely.*

The formula for radius of curvature when the equation of the curve is given in polar coördinates in the form $r = f(\theta)$, may be obtained in derivative form from § 64, (12). The form is,

$$(3) \qquad \rho = \frac{1}{k} = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\,\frac{d^2r}{d\theta^2}}.$$

The differential polar form for ρ may be obtained from this by multiplying both terms of the fraction by $d\theta^2$.

III. *The Centre of Curvature may be found as follows:* —

Let $OL = x'$ [fig. 32] and $LC = y'$ be the coördinates of C, the centre of curvature at $P(x, y)$ on the curve AB; whose equation may be given either in the form, $x = f_1(\phi)$, $y = f_2(\phi)$, or in the form, $y = f(x)$.

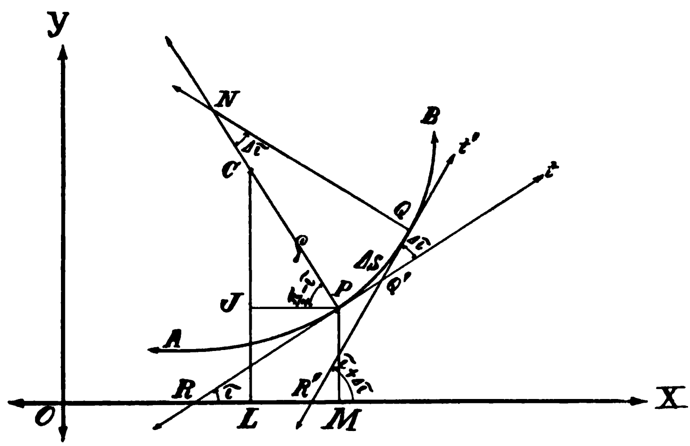

FIG. 32.

The figure shows that,

(4) $\quad x' = OL = OM - JP = x - JP,$

(5) $\quad y' = LC = MP + JC = y + JC$:

also that,

$$JP = PC \cos JPC = \rho \cos(90 - \tau) = \rho \sin\tau,$$
$$JC = PC \sin JPC = \rho \sin(90 - \tau) = \rho \cos\tau.$$

Substituting these values in (4) and (5) we obtain,

(6) $\quad x' = x - \rho \sin\tau,$

(7) $\quad y' = y + \rho \cos\tau.$

But, from equations (16) and (17) of § 64 we have,

(8) $$\sin\tau = \frac{dy}{ds};$$

(9) $$\cos\tau = \frac{dx}{ds}.$$

Substituting in (6) and (7) the values of $\sin\tau$ and $\cos\tau$, with the value of ρ from (1), we have, after reduction,

$$(10) \qquad x' = x - \frac{dy}{dx}\,\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{dx\,d^2y - dy\,d^2x}{dx^3}},$$

$$(11) \qquad y' = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{dx\,d^2y - dy\,d^2x}{dx^3}}.$$

Formulæ (10) and (11) will furnish the coördinates (x', y') of the centre of curvature corresponding to the point (x, y) on the curve whose equation is in the form, $x = f_1(\phi)$, $y = f_2(\phi)$; also, when the equation is in the form $y = f(x)$: but in the latter case $d^2x = 0$, and the formulæ take the following forms:—

$$(12) \qquad x' = x - \frac{dy}{dx}\,\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = x - \frac{(dx^2+dy^2)\,dy}{dx\,d^2y};$$

$$(13) \qquad y' = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{dx^2+dy^2}{d^2y}.$$

If (r', θ') are the polar coördinates of the centre of curvature at the point (r, θ) on the curve $r = f(\theta)$, we may obtain the values of $r'\cos\theta' = x'$ and $r'\sin\theta' = y'$ by substituting in (10) and (11) $r\cos\theta = x$ and $r\sin\theta = y$, as was done in § 64, VII. Putting ds for $\sqrt{(dr^2 + r^2d\theta^2)}$, we get from (10) and (11) after reducing,

$$(14) \qquad x' = r'\cos\theta' = r\cos\theta - \frac{(r\cos\theta\,d\theta + \sin\theta\,dr)\,ds^2}{(r^2d\theta^2 + 2dr^2 - r\,d^2r)\,d\theta},$$

$$(15) \qquad y' = r'\sin\theta' = r\sin\theta + \frac{(\cos\theta\,dr - r\sin\theta\,d\theta)\,ds^2}{(r^2d\theta^2 + 2dr^2 - r\,d^2r)\,d\theta}.$$

* *Exercises.*

Find the radius of curvature of each of the following curves: —

1. The *ellipse*, $b^2x^2 + a^2y^2 = a^2b^2$: $e^2 = \frac{1}{a^2}(a^2 - b^2)$.

$$Ans. \quad \rho = \frac{1}{k} = \frac{\mp (a^4 - e^2x^2)^{\frac{3}{2}}}{ab}.$$

2. The *hyperbola*, $b^2x^2 - a^2y^2 = a^2b^2$: $e^2 = \frac{1}{a^2}(a^2 + b^2)$.

$$Ans. \quad \rho = \frac{\mp (e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

3. The *cycloid*, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

$$Ans. \quad \rho = -4a\sin\frac{\phi}{2}.$$

4. The *involute of the circle*, $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$.

$$Ans. \quad \rho = a\phi.$$

5. The *semi-cubical parabola*, $ay^2 = x^3$.

$$Ans. \quad \rho = \frac{\pm x^{\frac{1}{2}}}{6a}(4a + 9x)^{\frac{3}{2}}.$$

6. The *astroid*, or *four-cusped hypocycloid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$Ans. \quad \rho = \pm 3(axy)^{\frac{1}{3}}.$$

Find the centre of curvature of each of the following curves: —

7. The *circle*, $x^2 + y^2 = a^2$.

$$Ans. \quad x' = y' = 0.$$

8. The *parabola*, $y^2 = 4px$.

$$Ans. \quad x' = 3x + 2p,\ y' = \frac{-y^3}{4p^2}.$$

9. The *ellipse*, $b^2x^2 + a^2y^2 = a^2b^2$.

$$Ans. \quad x' = \frac{(a^2 - b^2)x^3}{a^4},\quad y' = -\frac{(a^2 - b^2)y^3}{b^4}.$$

10. The *hyperbola*, $b^2x^2 - a^2y^2 = a^2b^2$.

$$Ans. \quad x' = \frac{(a^2 + b^2)x^3}{a^4},\quad y' = -\frac{(a^2 + b^2)y^3}{b^4}.$$

---

* *In the results of these exs. the minus sign belongs to that portion of the curve which is concave downwards; the plus sign to that portion which is concave upwards.*

11. The *cycloid*, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

Ans. $x' = a(\phi + \sin\phi)$, $y' = -a(1 - \cos\phi)$.

12. The *involute of the circle*, $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$.

Ans. $x' = a\cos\phi$, $y' = a\sin\phi$.

13. If a = radius of the fixed circle, b = radius of the rolling circle, and ϕ = angle between the x-axis and the line joining origin to centre of rolling circle, the equations of the epicycloid are,

$$\left\{\begin{aligned} x &= b\,[m\cos\phi - \cos m\phi] \\ y &= b\,[m\sin\phi - \sin m\phi] \end{aligned}\right\} : \quad m \equiv \frac{a+b}{b}.$$

Find,

$$\rho = \frac{1}{k} = \frac{4b(a+b)}{a+2b}\sin\frac{a\phi}{2b};$$

$$x' = \frac{ab}{a+2b}\Big[m\cos\phi + \cos m\phi\Big];$$

$$y' = \frac{ab}{a+2b}\Big[m\sin\phi + \sin m\phi\Big].$$

14. The corresponding results for the *hypocycloid* may be obtained by changing the sign of b throughout Ex. 13.

15. Given the polar equation of the *logarithmic spiral*, $r = e^{a\theta}$; find,

$$\left\{\begin{aligned} \rho &= \frac{1}{k} = r\sqrt{1+a^2}, \\ x' &= r'\cos\theta' = -ar\sin\theta = -ay, \\ y' &= r'\sin\theta' = ar\cos\theta = ax. \end{aligned}\right.$$

16. Given the polar equation of the *lemniscate*, $r^2 = a^2\cos 2\theta$; find,

$$\left\{\begin{aligned} \rho &= \frac{1}{k} = \frac{a^2}{3r}, \\ x' &= r'\cos\theta' = \frac{2a^2}{3r}\cos^3\theta, \\ y' &= r'\sin\theta' = \frac{-2a^2}{3r}\sin^3\theta. \end{aligned}\right.$$

CHAPTER XXVIII.

EVOLUTES AND INVOLUTES: A METHOD OF FINDING EVOLUTES OF PLANE CURVES: PROPERTIES OF THE EVOLUTE.

66. EVOLUTES OF PLANE CURVES.

Let $P(x, y)$ be a point on a continuous locus whose equation is either, (a) $x = f_1(\phi)$, $y = f_2(\phi)$, or (b) $y = f(x)$: and let $P'(x', y')$ be the centre of curvature corresponding to P.

Now, if P moves continuously on its curve, the point P' will trace a second curve; which is called **the Evolute of the first curve.** The first curve, the locus of P, is called **the Involute of the second**; that is: —

The Evolute, E, of a given curve, I, is the locus of the centre of curvature of I. Curve I is called the Involute of E.

To find the equation of the evolute, E, of a given curve, we must obtain the equation connecting x' and y' with the constants of the locus I by a process of elimination, which may be illustrated by the following example.

Example.

Find the evolute of the *parabola*, $y^2 = 4px$. From Ex. 8, § 65, we obtain,

$$x' = 3x + 2p, \text{ and } y' = \frac{-y^3}{4p^2}.$$

Solving these equations for x and y, we get,

(1) $$x = \frac{1}{3}(x' - 2p) \text{ and } y = -(4p^2y')^{\frac{1}{3}}.$$

But x and y are coördinates of a point on the given parabola: hence their values as shown in (1) must satisfy the equation of the parabola. Substituting them in $y^2 = 4px$ we get,

$$(4p^2y')^{\frac{2}{3}} = \frac{4p}{3}(x' - 2p).$$

This may be reduced to,

(2) $$y'^2 = \frac{4}{27p}(x' - 2p)^3.$$

Result (2) shows that the coördinates x', y' of the centre of curvature corresponding to a point P on the parabola, $y^2 = 4px$, will satisfy the equation,

(3) $$y^2 = \frac{4}{27p}(x - 2p)^3.$$

Hence, (3) *is the equation of the evolute of the parabola,* $y^2 = 4px$.

I. The foregoing example shows in outline the general method to be employed in finding the equation of the evolute, *when the equation of the involute is given in the form* $y = f(x)$. The method may be described as follows: —

(a) Find x' and y' by means of formulæ (12) and (13) of § 65;

(b) Obtain from these results the values of x and y in terms of x' and y';

(c) Substitute the values of x and y in the equation of the involute. *The result of this substitution will be the equation of the evolute, accents having been dropped from* x' *and* y' *after substitution.*

The method may be described more briefly as follows: —

Eliminate x and y from the following three equations: (1) the equation of the involute; (2) the value of x' in terms of x, etc., [Eq. (12), § 65]; and (3) the value of y' in terms of y, etc. [Eq. (13), § 65].

II. In the case *when the equation of the involute is in the form* $x = f_1(\phi)$, $y = f_2(\phi)$, we should employ formulæ (10) and (11) in finding x' and y'. These, usually, will give x' and y' in terms of ϕ; or can be reduced to such form. [Compare Exs. 11, 12, 13, of § 65.] The equation of the evolute may be obtained, in some cases, by eliminating ϕ from the two equations for x' and y'. For example, in Ex. 12, § 65, we may eliminate ϕ by using the equation $\sin^2\phi + \cos^2\phi = 1$; which gives $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{a}\right)^2 = 1$; whence *the evolute of the curve,* $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$, *is the circle,* $x^2 + y^2 = a^2$.

But the equations, $x = a\cos\phi$, $y = a\sin\phi$, represent the circle whose radius is a and centre is at origin: so that the elimination of ϕ is not necessary in this case, and sometimes it is extremely inconvenient. [See Ex. 13, § 65.]

III. ***When the equation of the involute is given in polar coördinates,*** **the equation of the evolute may be obtained by using formulæ (14) and (15) of § 65.**

It will be better, in general, to get x' and y' by (14) and (15); then to eliminate θ and r from the results, using the equation of the involute. If the equation of the evolute is desired in polar coördinates, a second transformation may be made. The following example will illustrate the method.

Example.

Find the evolute of the *lemniscate*, $r^2 = a^2 \cos 2\theta$.

Solution. **From Ex. 16, § 65, we obtain,**

$$(4) \qquad x' = \frac{2a^2}{3r}\cos^3\theta, \quad \text{and} \quad y' = \frac{-2a^2}{3r}\sin^3\theta.$$

This gives the equation of the evolute in the form, **$x = f_1(r, \theta)$, $y = f_2(r, \theta)$. We may eliminate r and θ from these equations by aid of $r^2 = a^2\cos 2\theta$. This can be done as follows:**

Solving (4), and dropping accents for convenience, we get,

$$\cos\theta = \left[\frac{3rx}{2a^2}\right]^{\frac{1}{3}}, \quad \text{and } \sin\theta = -\left[\frac{3ry}{2a^2}\right]^{\frac{1}{3}}.$$

$$(5) \quad \therefore \quad \cos^2\theta + \sin^2\theta = \left[\frac{3r}{2a^2}\right]^{\frac{2}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 1; \quad \text{and,}$$

$$(6) \qquad \cos^2\theta - \sin^2\theta = \left[\frac{3r}{2a^2}\right]^{\frac{2}{3}}(x^{\frac{2}{3}} - y^{\frac{2}{3}}) = \cos 2\theta = \frac{r^2}{a^2}.$$

From (5) and (6) we have,

$$\left[x^{\frac{2}{3}} + y^{\frac{2}{3}}\right]^2 = \left[\frac{2a^2}{3r}\right]^{\frac{4}{3}}, \quad x^{\frac{2}{3}} - y^{\frac{2}{3}} = \frac{r^2}{a^2}\left[\frac{2a^2}{3r}\right]^{\frac{2}{3}} = \frac{4a^2}{9}\left[\frac{3r}{2a^2}\right]^{\frac{4}{3}}.$$

Multiplying these results together, and reducing, we get,

$$(7) \qquad 9\,(x^{\frac{2}{3}} + y^{\frac{2}{3}})^2\,(x^{\frac{2}{3}} - y^{\frac{2}{3}}) = 4a^2;$$

which is the rectangular equation of the evolute of the lemniscate, $r^2 = a^2\cos 2\theta$.

The polar equation of the evolute may be obtained from (7) by the transformation, $x = r\cos\theta$, $y = r\sin\theta$.

67. TWO PROPERTIES OF EVOLUTES.

Let $P\ (x, y)$ be any point on the involute; and $Q\ (x', y')$ the corresponding point on the evolute. [See fig, 33.]

Now, the line through (x, y) and having the slope $-\frac{dx}{dy}$ is perpendicular to the tangent at (x, y); because the slope of the tangent is $\frac{dy}{dx}$. *Hence, the slope of the normal at* (x, y) *is* $-\frac{dx}{dy}$.

Also, the line through (x', y') having the slope $\frac{dy'}{dx'}$ will be tangent to the evolute at (x', y'); because the slope of the tangent at (x', y') is $\frac{dy'}{dx'}$, by § 11.

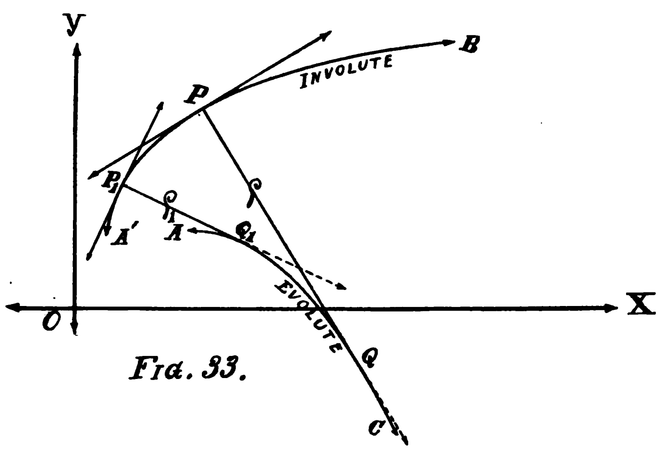

FIG. 33.

We wish to prove that:

I. *The normal of the involute at* $P(x, y)$ *is tangent to the evolute at the corresponding point* $Q(x', y')$.

In the first place, the normal at (x, y) passes through (x', y'), since (x', y') is the centre of curvature at (x, y) [see § 65, I]. We must prove, further, that this normal has the same slope as the tangent to the evolute at (x', y'); that is, we must prove that,

$$(1) \qquad \frac{dy'}{dx'} = -\frac{dx}{dy} = \frac{-1}{\frac{dy}{dx}}.$$

The relations between x', y', x, and y are given in equations (6) and (7) of § 65, III. They are,

$$(2) \qquad y' = y + \rho\cos\tau,$$
$$(3) \qquad x' = x - \rho\sin\tau.$$

The differentials of (2) and (3) are,

$$(4) \qquad dy' = dy + \cos\tau\, d\rho - \rho\sin\tau\, d\tau,$$
$$(5) \qquad dx' = dx - \sin\tau\, d\rho - \rho\cos\tau\, d\tau.$$

But, from § 64, (4), $k = \frac{1}{\rho} = \frac{d\tau}{ds}$: whence $ds = \rho\, d\tau$. Also, from § 64, (16) and (17), we have $dy = \sin\tau\, ds = \rho\sin\tau\, d\tau$; and $dx = \cos\tau\, ds = \rho\cos\tau\, d\tau$.

Put these values of dx and dy in (4) and (5) and we get, after canceling,

$$(6) \qquad dy' = \cos\tau\, d\rho,$$
$$(7) \qquad dx' = -\sin\tau\, d\rho.$$

Dividing (6) by (7) we get,

$$(8) \qquad \frac{dy'}{dx'} = -\operatorname{ctn}\tau = \frac{-1}{\tan\tau} = \frac{-1}{\frac{dy}{dx}};$$

since τ = angle between the x-axis and the tangent to the involute at (x, y). [See definition of τ in fig. 32].

Equation (8) being established, the theorem is proved.

It follows, therefore, that, if the point P traverses the involute, the normal at P will remain a tangent to the evolute; and its point of tangency is always the centre of curvature, Q, at the point P of the involute.

II. A second important property of the evolute is the following:

Let P_1 and P be two points on the involute, Q_1 and Q, the corresponding points on the evolute, and ρ_1, ρ the radii of curvature at P_1 and P; then the length of the arc, Q_1Q, of the evolute is equal to the difference, $\rho - \rho_1$, between the radii of curvature at P_1 and P.

Proof. We have shown above in (6) and (7) that $dx' = -\sin\tau\, d\rho$ and $dy' = \cos\tau\, d\rho$; where (x', y') is the point Q on the evolute, ρ is

the corresponding radius of curvature, and τ is the angle made with the x-axis by the tangent to the involute at the point P corresponding to Q [See Figs. 32 and 33].

Squaring, adding, and reducing these values for dx' and dy' we get,

$$(9) \qquad \sqrt{dx'^2 + dy'^2} = d\rho.$$

But the left member of (9) is the differential element of arc of the evolute [see § 57, (4)]. Call the length of the arc of the evolute, measured from a fixed point, A, to (x', y'), s': then $ds' = \sqrt{dx'^2 + dy'^2}$, and we have,

$$(10) \qquad ds' = d\rho, \text{ or } d\rho - ds' \equiv d(\rho - s') = 0;$$

$$(11) \therefore \qquad \frac{d(\rho - s')}{dx} = 0.$$

Since ρ and s' are both functions of x, and the derivative of $\rho - s'$ remains zero, it follows, from § 45, II, that $\rho - s' = l$, *where l is a constant, while Q traverses the arc Q_1Q.*

Now let s'_1 and ρ_1 be the values of s' and ρ at Q_1, and s' and ρ be their values at the second point, Q. Then $s'_1 = \rho_1 - l$ and $s' = \rho - l$; whence follows the theorem, viz.:

$$(12) \qquad \text{arc } Q_1Q = s' - s'_1 = (\rho - l) - (\rho_1 - l) = \rho - \rho_1.$$

III. From the two foregoing theorems it follows that, *if a stretched inelastic string be unwound from an evolute, its extremity will trace the involute.* For, the string will remain tangent to the evolute, and normal to the involute: and the length of string set free while tracing a given arc of the involute, will be equal to the length of the arc of the evolute which is thus uncovered.

Since the equation $\rho = s' + l$ involves the arbitrary constant l, it follows that we may obtain many involutes from one evolute by using different values of l. In fact, putting $s' = 0$ in $\rho - s' = l$, gives $l = \rho_0 =$ *the initial radius of curvature.*

Theorem II gives a method of finding the length of an arc of the evolute between two given points, by obtaining the difference between the radii of curvature at the two points.

Exercises.

Find the equations of the evolutes of the following involutes: —

1. Of the *ellipse*, $a^2y^2 + b^2x^2 = a^2b^2$.

Ans. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

2. Of the *hyperbola*, $a^2y^2 - b^2x^2 = -a^2b^2$.

Ans. $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.

3. Of the *cycloid*, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

Ans. $\left\{ \begin{matrix} x = a(\phi + \sin\phi) \\ y = -a(1 - \cos\phi) \end{matrix} \right\}$.

Plot this evolute.

4. Of the *astroid*, $x = a\cos^3\phi$, $y = a\sin^3\phi$.

Ans. $\left\{ \begin{matrix} x = a\cos^3\phi + 3a\cos\phi\sin^2\phi \\ y = 3a\cos^2\phi\sin\phi + a\sin^3\phi \end{matrix} \right\}$.

If ϕ is eliminated from this result we get, $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$ for the rectangular equation of the evolute: and if the rectangular axes are turned through an angle of 45°, this last result becomes $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$.

5. Of the *logarithmic spiral*, $r = e^{a\theta}$.

Ans. $\left\{ \begin{matrix} x = -ar\sin\theta \\ y = ar\cos\theta \end{matrix} \right\}$.

If θ and r are eliminated between this result and $r = e^{a\theta}$, we get for the rectangular equation of the evolute,

$$x^2 + y^2 = a^2 e^{-2a\tan^{-1}\frac{x}{y}}.$$

Transforming the last result to polar coördinates gives,

$$r = \pm\, a e^{a\left(\theta - \frac{\pi}{2}\right)},$$

which is the polar equation of the evolute of $r = e^{a\theta}$.

CHAPTER XXIX.

SOME METHODS OF APPROXIMATE INTEGRATION.

TRAPEZOIDAL RULES: SIMPSON'S RULE: BY SERIES.

68. APPROXIMATE INTEGRATION.

It has been shown in § 33 that the integral,

$$(1) \qquad G = \int_a^b f(x)\,dx,$$

can be evaluated, in general, when we are able to find the anti-differential of $f(x)\,dx$. Now the problem of anti-differentiation is often unsolvable (see § 27), and is frequently so difficult as to make it worth while to possess simple methods of obtaining an approximate value of G without recourse to anti-differentiation. Such methods will be explained below. Each is based on the fact that the integral, G, can be geometrically represented as an area under the locus of the equation, $y = f(x)$; and, hence, an approximate value of this area will be an equally close approximate value of G.

A. Trapezoidal Methods.

Let the curve AB, in fig. 34, be the locus of the equation,

$$(2) \qquad y = f(x).$$

Let $OM_0 = a$, and $OM_{2n} = b$; then the area under the arc P_0P_{2n} will represent the value of G in (1). Divide the interval $M_0M_{2n} = b - a$ into $2n$ equal parts, and call each part h; then,

$$(3) \qquad M_0M_1 = M_1M_2 = \ldots\ldots = \frac{b-a}{2n} = h.$$

Draw the ordinates, $y_0, y_1, y_2, y_3, \ldots\ldots y_{2n}$; and draw the chords, P_0P_1, P_1P_2, etc., through the tops of the ordinates. This will

form a set of $2n$ trapezoids inscribed under the arc; and the sum of their areas will be an approximate value of the area under the curve. The area of $M_0P_0P_1M_1$ is, by geometry, $\frac{1}{2}(y_0 + y_1)h$. Forming the expression for each of the areas of the $2n$ trapezoids, and taking their sum, we get,

$$(4) \quad area = h\left[\tfrac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + y_{2n-1} + \tfrac{1}{2}y_{2n}\right]$$

$$= \int_a^b f(x)\,dx, \textit{ approximately.}$$

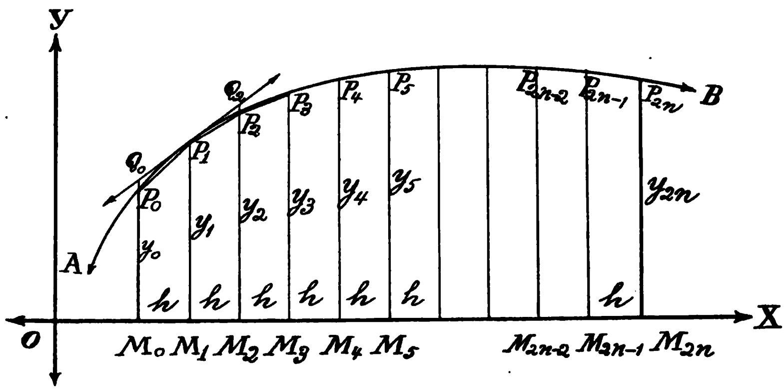

FIG. 34.

The ordinates, y_0, y_1, y_2, etc., in (4), are obtained directly from (2) by substituting for x, in succession, the abscissæ of the points P_0, P_1, P_2, etc. These abscissæ are a, $a + h$, $a + 2h$, etc.

In the above, the number of divisions of the interval $b - a$ is taken an even number, $2n$; *but it could as well be taken an odd number.*

If the curve, AB, is concave downwards, as in fig. 34, formula (4) will give a result smaller than G; but if the curve is concave upwards, the result obtained for G by using (4) will be too large.

Obviously, in general, the approximation to G will be made closer by increasing n.

Example.

Find an approximate value for $\int_1^5 \frac{dx}{x} = \log 5$.

Taking $2n = 12$, we get $h = \frac{5-1}{12} = \frac{1}{3}$; and the successive values of x are, $1, 1 + h = \frac{4}{3}, 1 + 2h = \frac{5}{3}, 1 + 3h = 2, \ldots, 1 + (2n-1)h = \frac{14}{3}$, and $1 + 2nh = 5$. The corresponding values of y are the reciprocals of the values of x; since $y = f(x) = \frac{1}{x}$.

Substituting in (4) we get,

$$(5) \qquad area = \frac{1}{3}\left[\frac{1}{2} + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \frac{3}{7} + \cdots + \frac{3}{13} + \frac{3}{14} + \frac{1}{10}\right]$$

$$= 1.61823 = \int_1^5 \frac{dx}{x} = \log 5, \textit{ approximately.}$$

The curve $y = \frac{1}{x}$ *is concave upwards from* $x = 1$ *to* $x = 5$; *hence our result for* $\log 5$ *is too large.*

A second trapezoidal formula can be obtained by using fig. 34 and drawing tangents to the curve at the ends of the alternate ordinates $y_1, y_3, y_5, \ldots, y_{2n-1}$. If the ordinates, y_0, y_2, etc., up to y_{2n}, are produced (when necessary, — that is, when the curve is concave downwards) to meet these tangents, a set of circumscribed trapezoids will be formed. The trapezoid $M_0 Q_0 Q_2 M_2$ is one of them; and its area is $2hy_1$.

The student can readily obtain the following expression for the sum of the areas of these circumscribed trapezoids:

$$(6) \qquad area = 2h[y_1 + y_3 + y_5 + y_7 + \cdots + y_{2n-1}]$$

$$= \int_a^b f(x)\,dx, \textit{ approximately.}$$

This formula requires that the number of divisions of the interval, $b - a$, *shall be even.*

It may be seen that, in general, (6) will give too large a result when (4) gives a result too small; and *vice versa*.

It is plain, from fig. 34, that the mean of the results (4) and (6) will be a closer approximation to the value of the integral, G, than either (4) or (6).

If we add (4) and (6) and divide the result by 2, we get the following mean of (4) and (6):

$$(7) \quad area = \frac{h}{2}\left[\frac{1}{2}y_0 + y_2 + y_4 + y_6 + \cdots + y_{2n-2} + 3(y_1 + y_3 + y_5 + \cdots + y_{2n-1}) + \frac{1}{2}y_{2n}\right]$$

$$= \int_a^b f(x)\,dx, \text{ approximately.}$$

This formula, like (6), requires that the number of divisions shall be even.

Exercises.

1. Compute approximate values of $\int_1^5 \frac{dx}{x} = \log 5$, taking $2n = 12$, and using (6), then (7). Compare the results with each other and with the value of $\log 5 = 1.609438$, — which is its correct value to six decimal places.

2. Compute $\int_1^{17} \frac{dx}{x} = \log 17$, approximately, taking $2n = 8$ and using each of the three trapezoidal formulæ, (4), (6), and (7). Compare each with the correct value, $\log 17 = 2.833213$.

3. Compute $4\int_0^1 \frac{dx}{1+x^2} = 4\tan^{-1}1 = \pi$, approximately, taking $2n = 10$ and using (4), (6), and (7). Compare each result with the correct value, $\pi = 3.14159265$.

B. Simpson's Method, or Rule.

This method is based on the following propositions: —

(*a*) *The equation, $y = ax^2 + bx + c$, has for its locus a parabola whose axis is parallel to the y-axis; and the arbitrary constants, a, b, c, may be so chosen as to make the parabola pass through any three given points.*

(*b*) *If three points, P, Q, R, are taken so that the ordinates of P and R are equidistant from Q, — that is, the coördinates of P, Q, and R, are $P(x' - h, y')$, $Q(x', y'')$ and $R(x' + h, y''')$, — then, the area under the arc, PQR, of the parabola is,*

$$(8) \qquad \text{area} = \frac{h}{3}(y' + 4y'' + y''').$$

The first proposition will be evident if we transform to new coördinate axes, parallel to the old, with the new origin at the point $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$. The equation becomes $x^2 = \frac{1}{a}y$, — which is the familiar form of the equation of a parabola whose vertex is at the origin and whose axis coincides with the y-axis. Hence $y = ax^2 + bx + c$ is represented by a parabola whose vertex is $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$, and whose axis is parallel to the y-axis.

The second proposition may be established as follows: —

The area under the curve, $y = ax^2 + bx + c$, from $x = x' - h$ to $x' + h$ is,

$$(9) \qquad \int_{x'-h}^{x'+h}(ax^2 + bx + c)\,dx = h\left[\frac{a}{3}(6x'^2 + 2h^2) + 2bx' + 2c\right].$$

If the points $P(x' - h, y')$, $Q(x', y'')$, and $R(x' + h, y''')$ are on the parabola, $y = ax^2 + bx + c$, then we must have,

$$\left\{\begin{aligned} y' &= a(x' - h)^2 + b(x' - h) + c \\ y'' &= ax'^2 + bx' + c \\ y''' &= a(x' + h)^2 + b(x' + h) + c \end{aligned}\right\}.$$

Solving these equations for a, b, and c, we get,

$$a = \frac{1}{2h^2}(y' - 2y'' + y'''),$$

$$b = \frac{1}{2h}(y''' - y') - \frac{x'}{h^2}(y' - 2y'' + y'''),$$

$$c = y'' - \frac{x'}{2h}(y''' - y') + \frac{x'^2}{2h^2}(y' - 2y'' + y''').$$

Putting these values of a, b, and c in (9), and reducing, we get the area under the arc PQR to be $\frac{h}{3}(y' + 4y'' + y''')$, — which is the value given in equation (8). *This establishes proposition* (b).

Turning now to fig. 34, conceive a parabola, with its axis parallel to OY, to be passed through the three points P_0, P_1, and P_2. The parabolic arc, $P_0\,P_1\,P_2$, will more nearly coincide with the arc $P_0\,P_1\,P_2$ of the curve AB than does the broken line $P_0\,P_1\,P_2$. Hence, the area under the parabolic arc $P_0\,P_1\,P_2$ will be a closer approximation to the area under the arc $P_0\,P_1\,P_2$ of the curve AB than is the sum of the areas of the two trapezoids under the broken line $P_0\,P_1\,P_2$.

In like manner, the area under the parabolic arc through $P_2\,P_3\,P_4$ will be a good approximation to the area under the arc $P_2\,P_3\,P_4$ of the curve AB. And so on, till we finally reach the arc $P_{2n-2}\,P_{2n-1}\,P_{2n}$.

Now, using (8), we readily obtain the following set of equations : —

Area under $P_0\,P_1\,P_2 = \frac{h}{3}(y_0 + 4y_1 + y_2)$, *approximately.*

" " $P_2\,P_3\,P_4 = \frac{h}{3}(y_2 + 4y_3 + y_4)$, "

" " $P_4\,P_5\,P_6 = \frac{h}{3}(y_4 + 4y_5 + y_6)$, "

. . . .

" " $P_{2n-2}\,P_{2n-1}\,P_{2n} = \frac{h}{3}(y_{2n-2} + 4y_{2n-1} + y_{2n})$, " .

Adding these results we obtain,

$$(10)\quad \textit{area under arc } P_0 P_{2n} = \int_a^b f(x)\, dx$$

$$= \frac{h}{3}\Big[y_0 + 2\,(y_2 + y_4 + y_6 + \ldots. + y_{2n-2})$$

$$+ 4\,(y_1 + y_3 + y_5 + \ldots. + y_{2n-1}) + y_{2n}\Big],$$

approximately.

Formula (10) is known as **Simpson's Rule**, or method, of approximate integration. *It requires an even number of divisions of* $b - a$.

This formula is more troublesome in application than either of the trapezoidal formulæ; but it will furnish an approximation considerably closer than either (4) or (6); and in most cases, probably, closer than (7).

C. By convergent series.

When the function, $f(x)$, can be expanded into a convergent series, the integral, $\int_a^b f(x)\, dx$, can be found approximately by integrating the series term by term (see Exs. 11, 13, p. 143).

Example.

Take $2n = 12$ and we readily get,

$$\int_1^5 \frac{dx}{x} = \frac{1}{9}\Big[1 + 3 + \frac{6}{5} + 2 + \frac{6}{7} + \frac{3}{2} + \frac{2}{3} + \frac{6}{5} + \frac{6}{11} + 1 + \frac{6}{13} + \frac{6}{7} + \frac{1}{5}\Big]$$

$$= 1.6097717 = \log 5, \textit{ approximately.}$$

Exercises.

1. Solve again Exs. 1, 2, and 3 on p. 187, using ***Simpson's Rule***, and compare results with those obtained by the trapezoidal rules.

2. Compute $\int_{10}^{20} \log_{10} x\,dx = 11.677655$, taking $2n = 10$ and using Simpson's rule.

3. Compute $\int_{0}^{1.2} \frac{dx}{1+x^2} = 0.9218$, taking $2n = 6$ and using Simpson's rule.

4. If $l =$ *length* of a simple pendulum which oscillates through an angle $\alpha < \pi$ on each side of the vertical, and $t =$ *time in seconds* of a half oscillation (*from one highest point to the other*), then it is shown in mechanics that,

$$t = 2\sqrt{\frac{l}{g}}\int_{0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \text{ where } k = \sin\frac{1}{2}\alpha.$$

Obtain the following series for t:

$$t = \pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}k\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}k^2\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^3\right)^2 + \cdots\right].$$

5. Given $g = 32.2$, $l = 12$ *inches*, and $\alpha = 15°$, find t.

CHAPTER XXX.

PARTIAL DIFFERENTIATION: APPLICATIONS TO SOLID GEOMETRY.

69. PARTIAL DIFFERENTIATION.

I. It is shown in solid analytic geometry that the locus of the equation $x^2 + y^2 + z^2 - r^2 = 0$, is the sphere whose centre is at the origin and whose radius is r: also that $x = a$, $y = b$, and $z = c$ are the equations of planes perpendicular, respectively, to the x-, y-, and * z-axes. Solving the equation of the sphere for z we get,

(1) $$z = \pm \sqrt{r^2 - x^2 - y^2}.$$

If we give to x and y the values a and b, where $a < r$ and $b < r$, two values of z are fixed by (1), which represent the perpendicular distances of the surface of the sphere from the point (a, b) in the xy-plane.

If we give to y the value b, $< r$, but do not assign a value to x, then (1) will not determine z; and z will be a function of x. Indeed, the combined equations, $y = b$ and $z = \pm \sqrt{r^2 - x^2 - y^2}$, will be represented by the circle in which the plane $y = b$ cuts the sphere.

In like manner we may give to x a value x, $< r$, and z will then be a function of y.

It is plain that we may vary z in three ways: (1) *by varying x, keeping y fixed;* (2) *by varying y, keeping x fixed; or* (3) *by varying both x and y.* It is plain also that fixing values of x and y will fix two corresponding values for z; and that z cannot be fixed unless both x and y are fixed.

In such a case we call z a function of the two independent variables, x and y.

General symbols for such functions are, $z = f(x, y)$, $z = \phi(x, y)$, etc.

---

* The axes are three mutually perpendicular lines, which determine three mutually perpendicular coördinate planes, called the xy-, xz-, and yz-planes.

II. Let us consider a function,

$$(2) \qquad z = f(x, y),$$

of two independent variables, x and y. Give to x the increment Δx, keeping y fixed. Then z will take an increment (which we shall represent by the symbol $\Delta_x z$) which will be given by the formula,

$$(3) \qquad \Delta_x z = f(x + \Delta x, y) - f(x, y).$$

Divide (3) by Δx and we get,

$$(4) \qquad \frac{\Delta_x z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

If the ratio in (4) approaches a limit when $\Delta x \doteq 0$, we shall call that limit **the partial derivative of z with respect to x**; *and shall denote it by the symbol,** $\frac{\partial z}{\partial x}$; that is,

$$(5) \qquad \frac{\partial z}{\partial x} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta_x z}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\right].$$

In like manner,

$$(6) \qquad \frac{\partial z}{\partial y} = \lim_{\Delta y \doteq 0}\left[\frac{\Delta_y z}{\Delta y}\right] = \lim_{\Delta y \doteq 0}\left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\right],$$

is the partial derivative of $z = f(x, y)$ with respect to y. It is derived on the hypothesis that x is a constant for the time being, and y is independent variable.

Partial derivatives, as defined in (5) and (6), are calculated for the ordinary functions by the same formulæ as the ordinary derivatives which are defined in § 9.

For example, if $z = ax + by + c$, $\frac{\partial z}{\partial x} = a$, and $\frac{\partial z}{\partial y} = b$. If $z = \sin(x + y)$, $\frac{\partial z}{\partial x} = \cos(x + y)$, and $\frac{\partial z}{\partial y} = \cos(x + y)$.

If $z = \sqrt{r^2 - x^2 - y^2}$, $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{r^2 - x^2 - y^2}}$, and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{r^2 - x^2 - y^2}}$.

---

* This symbol may be read: *Round-d-z over round-d-x.*

III. **The partial differentials** *of z with respect to x and y*, may be defined by the aid of (5) and (6). If we denote the partial differentials of z, respectively, by $\partial_x z$ and $\partial_y z$ we have, as definitions,

$$\partial_x z = \frac{\partial z}{\partial x}\,dx, \qquad \partial_y z = \frac{\partial z}{\partial y}\,dy. \tag{7}$$

Here dx and dy are the ordinary differentials of x and y. The first equation in (7) defines the partial differential of z *with respect to x*; the second defines the partial differential of z *with respect to y*.

The *sum of the partial differentials is called* **the total differential of** z. Denoting it by dz we have, as a definition,

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{8}$$

Dividing this *total differential of z* (1) by dx, (2) by dy, we have,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}, \tag{9}$$

$$\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y}. \tag{10}$$

Equation (9) will define **the total derivative of** z *with respect to x*, and (10) will define **the total derivative of** z *with respect to y*, when $z = f(x, y)$.

IV. If $z = f(x, y) = 0$, then $f(x, y) = 0$ presents y as an implicit function of x (see § 2), and $\frac{dz}{dx} = 0$. In this case we may put the right member of (9) in the form,

$$\frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} = 0. \tag{11}$$

Whence,

$$\frac{dy}{dx} = -\frac{\partial f(x, y)}{\partial x} \div \frac{\partial f(x, y)}{\partial y}. \tag{12}$$

Equation (12) is a general formula for the derivative of y with respect to x, *when y is an implicit function of x*. For example, if $f(x, y) = \sin(x + y) + \log x - e^y = 0$, we get by using (12),

$$\frac{dy}{dx} = -\left[\cos(x + y) + \frac{1}{x}\right] \div \left[\cos(x + y) - e^y\right].$$

Formula (12) will enable us to find the slope, $\frac{dy}{dx}$, of any curve $f(x, y) = 0$ even when we can not solve the equation for either x or y. [*Compare* §§ 9 *and* 11.]

Exercises.

1. Verify the general formula $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, where $f \equiv f(x, y) = 0$, for the following expressions: –

(*a*) $f \equiv ax^2 + by^2 + 2cxy + 2dx + 2ey + g = 0$;

(*b*) $f \equiv \sin x \log y + \cos y \, e^{\sin x}$.

2. Find $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ for the two expressions (*a*) and (*b*) in Ex. 1.

3. Find dz, $\frac{dz}{dx}$, and $\frac{dz}{dy}$, when,

(*a*) $z = \frac{a^3}{xy}$, or $xyz = a^3$;

(*b*) $z = \sqrt{x^2 + y^2}$;

(*c*) $z = e^x \log \sin y$;

(*d*) $z = \tan^{-1} \frac{y}{x}$.

V. If $w = f(x, y, z)$ is a function of three independent variables, we may define (as in II and III above) *the partial derivatives and partial differentials* of w with respect to x, y, and z. For *the partial differentials of* w we should have,

$$(13)^* \quad \partial_x w = \frac{\partial f}{\partial x} dx, \; \partial_y w = \frac{\partial f}{\partial y} dy, \; \partial_z w = \frac{\partial f}{\partial z} dz.$$

For *the total differential of* w we have,

$$(14)^* \quad dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The formulæ for the three total derivatives of w *with respect to* x, y, *and* z, *respectively, may be formed from* (14) *by dividing successively by* dx, dy, *and* dz.

---

* The $\frac{\partial f}{\partial x}$, etc., in (13) and (14) are abbreviations for $\frac{\partial f(x, y, z)}{\partial x}$, etc.

If $w = f(x, y, z) = 0$, then dw in (14) is zero.

A notation for *the partial derivatives of* $f \equiv f(x, y, z) = 0$, which is sometimes very convenient, is the following: —

Denote $\frac{\partial f}{\partial x}$ by f'_x, $\frac{\partial f}{\partial y}$ by f'_y, and $\frac{\partial f}{\partial z}$ by f'_z.

We may then use the symbols f'_{x_1}, f'_{y_1}, f'_{z_1} *to denote the values of the partial derivatives of* $f(x, y, z) = 0$ *at the point* (x_1, y_1, z_1);

that is, $f'_{x_1} \equiv \left.\frac{\partial f(x, y, z)}{\partial x}\right|_{x = x_1}$, etc.

Exercises.

1. The equation $f(x, y, z) \equiv Ax + By + Cz + D = 0$, in rectangular space coördinates, represents a plane. The normal to the plane from the origin makes the angles α, β, and γ, with the x-, y-, and z-axes. Let $\lambda = \cos\alpha$, $\mu = \cos\beta$, and $\nu = \cos\gamma$ be the *direction cosines* of this normal. It is shown in solid analytic geometry that,

$$\lambda = A \div H,\ \mu = B \div H, \text{ and } \nu = C \div H, \text{ where } H = \pm\sqrt{(A^2 + B^2 + C^2)}.$$

Prove that,

$$\lambda = f'_x \div J,\ \mu = f'_y \div J, \text{ and } \nu = f'_z \div J,$$

where

$$J = \pm\sqrt{\left(f'^2_x + f'^2_y + f'^2_z\right)}.$$

2. Given that the plane,

$$(x - x_1)f'_{x_1} + (y - y_1)f'_{y_1} + (z - z_1)f'_{z_1} = 0,$$

is tangent to the surface $f(x, y, z) = 0$ at the point P_1. Find the equations of the tangent planes of the following surfaces at the point P_1: —

(*a*) The *sphere*, $x^2 + y^2 + z^2 - a^2 = 0$.

Ans. $x_1x + y_1y + z_1z - a^2 = 0$.

(*b*) The *ellipsoid*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans. $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} + \frac{z_1z}{c^2} = 1$.

3. Find the *direction-cosines* of the normal to each of the tangent planes found in Ex. 2. *These will, obviously, be the direction-cosines of the surface at the point* P_1.

4. Given that the equations of the normal at P_1 to the surface $f(x, y, z) = 0$ are,

$$\frac{x - x_1}{f'_{x_1}} = \frac{y - y_1}{f'_{y_1}} = \frac{z - z_1}{f'_{z_1}}:$$

find the equations of the normals at P_1 to the surfaces whose equations are given in Ex. 2.

5. The locus of two simultaneous equations, $\left\{\begin{matrix} f(x, y, z) = 0 \\ \phi(x, y, z) = 0 \end{matrix}\right\}$, *is the curve of intersection of the two surfaces* $f(x, y, z) = 0$ *and* $\phi(x, y, z) = 0$.

Given that the equations of the line tangent to this curve at the point P_1 are,

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N};$$

and that the equation of the plane through P_1 (called *the normal plane*) perpendicular to the tangent is,

$$(x - x_1) L + (y - y_1) M + (z - z_1) N = 0; \text{ where}$$

$$L \equiv f'_{y_1} \phi'_{z_1} - f'_{z_1} \phi'_{y_1}, \quad M \equiv f'_{z_1} \phi'_{x_1} - f'_{x_1} \phi'_{z_1}, \quad N \equiv f'_{x_1} \phi'_{y_1} - f'_{y_1} \phi'_{x_1}.$$

Show that the equations of the line tangent at P_1 to the curve in which the *sphere* $x^2 + y^2 + z^2 = r^2$ is cut by the *cylinder*, $x^2 + y^2 = rx$, are,

$$\left\{\begin{matrix} r(x - x_1) + 2z_1(z - z_1) = 0 \\ ry_1(y - y_1) + z_1(r - 2x_1)(z - z_1) = 0 \end{matrix}\right\};$$

and show that the equation of the normal plane at P_1 is,

$$2y_1 z_1 x + z_1(r - 2x_1) y + r y_1 z = 0.$$

6. The equations of *the helix* are,

$$\left\{\begin{matrix} x^2 + y^2 - r^2 = 0 \\ y - x \tan \frac{z}{c} = 0 \end{matrix}\right\}.$$

Show that the equations of *the tangent line* are,

$$\left\{\begin{matrix} c(x - x_1) + y_1(z - z_1) = 0 \\ c(y - y_1) - x_1(z - z_1) = 0 \end{matrix}\right\};$$

and that the equation of *the normal plane* is,

$$y_1 x - x_1 y - c(z - z_1) = 0.$$

7. Find the equations of the tangent to the circle in which *the plane*, $ax + by + cz - 1 = 0$, cuts *the sphere*, $x^2 + y^2 + z^2 - r^2 = 0$, at the point P_1.

8. Find the equations of the tangent to the ellipse in which *the plane*, $ax + by + cz - 1 = 0$, cuts *the cylinder*, $x^2 + y^2 - r^2 = 0$.

9. Given that the differential, ds, of the length, s, of a space curve is,

$$ds = \sqrt{dx^2 + dy^2 + dz^2};$$

find the lengths of the following curves: —

(*a*) The helix, $x = r\cos\phi$, $y = r\sin\phi$, $z = a\phi$.

(*b*) The curve, $x^2 = 2py$, $x^2 = 2qz$.

(*c*) The curve, $x^2 + y^2 + z^2 = r^2$, $x^2 + y^2 = rx$.

[Suggestion. *In (c) transform to polar coördinates by the formulæ,* $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, *and* $z = r\cos\theta$.

10. Given that the direction-cosines at (x, y, z) of a space curve are,

$$\cos\alpha = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds};$$

find the direction-cosines of *the helix*, $x = r\cos\phi$, $y = r\sin\phi$, $z = a\phi$.

CHAPTER XXXI.

MULTIPLE INTEGRATION: APPLICATIONS TO AREAS, SURFACES, AND VOLUMES: CENTRES OF GRAVITY: MOMENTS OF INERTIA.

70. DOUBLE INTEGRATION: APPLICATIONS.

I. The area of the rectangle, $M_k Q$, in fig. 35 may be obtained by a summation, or integration, similar to that explained in §§ 31 and 33. It is as follows: —

Divide the altitude $M_k P_k, = y_k$, into m equal parts, of which $AD, = y_k \div m$, is one. Call it dy. Then $\lim\limits_{m=\infty} [dy] = 0$. Construct the m rectangles on the divisions of y_k, — $ABCD$ is one of

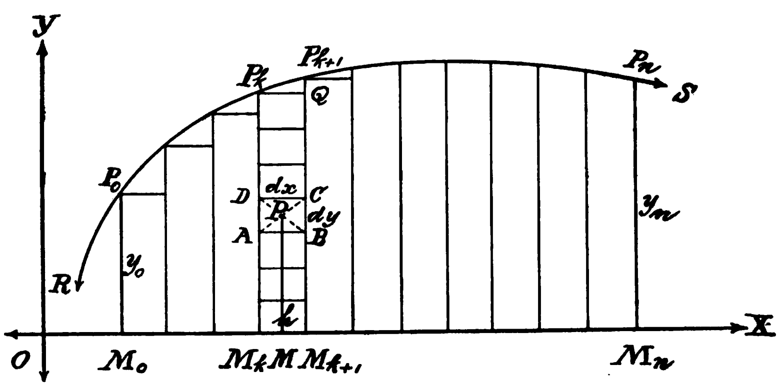

FIG. 35.

them. Call the common base of these rectangles, h. Then, area $ABCD = AB \times AD = h\,dy$; and the area of the rectangle $M_k Q$ is,

(1) $$M_k Q = \lim_{m=\infty} \sum_0^{y_k} h\,dy = \int_0^{y_k} h\,dy = hy_k.$$

In this, y is independent variable and h is constant.

Let us consider the area under the arc $P_0 P_n$ of the curve RS whose equation is $y = f(x)$. Let $OM_0 = a$, $OM_n = b$, and $OM_k = x_k$. Then $y_k = f(x_k) = M_k P_k$.

We have shown in § 31 that the area under $P_0 P_n$ is the limit of the sum of n rectangles inscribed under $P_0 P_n$. Let $M_k Q$ be one of the n rectangles. Then the base is $M_k M_{k+1} = \dfrac{M_0 M_n}{n} = \dfrac{b-a}{n} = dx$; and this is equal to the h in equation (1). Replacing h by dx, we may express the area under $P_0 P_n$, as shown in § 31, by the formula,

$$(2) \qquad G = \lim_{n=\infty} \sum_a^b M_k Q = \lim_{n=\infty} \sum_a^b \left[\lim_{m=\infty} \sum_0^y dx\,dy \right]$$
$$= \int_a^b \left[\int_0^y dx\,dy \right] \equiv \int_a^b \int_0^y dx\,dy.$$

This is a *double summation*, or **double integration**; *first, with respect to* y as independent variable, in which operation dx is a constant as the h is in (1); and, *secondly, with respect to* x. The result expresses the area under the curve $y = f(x)$, between the parallel lines $x = a$ and $x = b$. Formula (2) is called a **double integral.**

The identity of formula (2) with formula (16) of § 33, p. 70 may be seen by noting that if dx is a constant, for the time being, we have,

$$(3) \qquad G = \int_a^b \int_0^y dx\,dy \equiv \int_a^b \left[\int_0^y dy \right] dx = \int_a^b y\,dx;$$

since $\displaystyle\int_0^y dy = y$.

Hence, (2) may be obtained from $\displaystyle\int_a^b y\,dx$ by putting $\displaystyle\int_0^y dy$ for y.

In using (2) we must replace y by $f(x)$ in the second integration; or, we may put (2) in the form,

$$(4) \qquad G = \int_a^b \int_0^{f(x)} dx\,dy.$$

II. If the area, G, is bounded by two curves, $y = \phi(x)$ and $y = f(x)$, and the two parallel lines $x = a$ and $x = b$, we should get,

$$(5) \qquad G = \int_a^b \int_{\phi(x)}^{f(x)} dx\, dy.$$

In short, (4) is the special case of (5) in which the lower boundary of the area is the line $y = \phi(x) = 0$, or the x-axis.

III. If the area is included between two curves, $x = \phi(y)$ and $x = f(y)$, and two parallel lines, $y = c$ and $y = d$, we should get the following double integral value for the area:

$$(6) \qquad G = \int_c^d \int_{\phi(y)}^{f(y)} dy\, dx.$$

If the area is included between the y-axis, the curve $x = f(y)$, and the parallels $y = c$ and $y = d$ we get,

$$(7) \qquad G = \int_c^d \int_0^{f(y)} dy\, dx.$$

IV. A double-integral formula may be obtained for areas in polar coördinates; for (see fig. 30, p. 151) the area of the sector, OPR, is $\frac{1}{2} r^2 d\theta = \int_0^r r\, d\theta\, dr$, in which $d\theta$ is a constant. Then, the whole area OP_0P_n is,

$$(8) \qquad G = \int_{\theta_0}^{\theta_n} \int_0^r r\, d\theta\, dr = \int_{\theta_0}^{\theta_n} \int_0^{f(\theta)} r\, d\theta\, dr;$$

when $r = f(\theta)$ is the equation of the curve AB in fig. 30.

V. Double-integral formulæ for the volume, and for the surface, generated by rotating the curve $y = f(x)$ about the x-axis, may be obtained from (5), p. 91, and (4), p. 149. These are,

$$(9) \qquad E = 2\pi \int_a^b \int_0^{f(x)} y\, dx\, dy \equiv \pi \int_a^b y^2 dx,$$

for the volume generated by the area under $y = f(x)$ between $x = a$ and $x = b$; and,

$$(10) \qquad F = 2\pi \int_a^b \int_0^{f(x)} ds\, dy = 2\pi \int_a^b \int_0^{f(x)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\, dy,$$

for the area of the surface generated by rotating $y = f(x)$ about the x-axis.

If the curve $x = f(y)$ is rotated about the y-axis, we get,

$$(11) \qquad E = 2\pi \int_c^d \int_0^{f(y)} x\, dy\, dx \equiv \pi \int_c^d x^2 dy;$$

$$(12) \qquad F = 2\pi \int_c^d \int_0^{f(y)} ds\, dx = 2\pi \int_c^d \int_0^{f(y)} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy\, dx;$$

for the volume, E, and the area, F, of the surface generated.

The student has noticed already, no doubt, that double-integration is a **partial integration**, — analogous to partial differentiation, in that one variable is, for the time being, a constant.

Exercises.

1. Find by double integration the areas of the following curves: —

(*a*) The *circle*, $x^2 + y^2 = a^2$;

(*b*) The *ellipse*, $b^2 x^2 + a^2 y^2 = a^2 b^2$;

(*c*) The *astroid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$;

(*d*) The *lemniscate*, $r^2 = a^2 \cos 2\theta$;

(*e*) The *cardioid*, $r = a(1 - \cos\theta)$.

2. Find the volume, and the surface, generated by rotating about the x-axis one arch of *the companion to the cycloid*,

$$\begin{Bmatrix} x = a\phi \\ y = a(1 - \cos\phi) \end{Bmatrix}.$$

3. Show that the volume of *the sphere*, $x^2 + y^2 + z^2 = a^2$, is

$$V = 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \frac{4}{3} \pi a^3.$$

4. Show that the volume of *the ellipsoid*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is,

$$V = 8c\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dx\, dy = \frac{4}{3}\pi abc.$$

5. Show that the volume cut off *the elliptic paraboloid*, $\frac{y^2}{b} + \frac{z^2}{c} = 2x$, by the plane $x = a$ is,

$$V = 4\sqrt{\frac{c}{b}} \int_0^a \int_0^{\sqrt{2bx}} \sqrt{2bx - y^2}\, dx\, dy = \pi a^2 \sqrt{bc}.$$

6. Show that each of the integrals in Exs. 3, 4, and 5 can be expressed in the form of a triple integral,

$$Vol. = K\int_a^b \int_0^{f(x)} \int_0^{\phi(x,y)} dx\, dy\, dz.$$

Find the limits of each integration. Illustrate with a figure the geometric significance of the three successive summations effected by the three successive integrations in each problem.

7. Find the volume enclosed by the *xy-plane*, the *plane*, $z = mx$, and the *cylinder*, $x^2 + y^2 = a^2$.

Ans. $V = \frac{2}{3}a^3 m.$

8. Find the volume common to the two cylinders, $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Ans. $V = \frac{16a^3}{3}.$

9. Find the volume common to the *paraboloid of rotation*, $y^2 + z^2 = 4ax$ and the *cylinder*, $x^2 + y^2 = 2ax$.

Ans. $V = \left(2\pi + \frac{16}{3}\right)a^3.$

10. Given that, when $z = f(x, y)$ is the equation of a surface (*axes rectangular*), the integral formula for area of surface is,

$$S = \int_{x_0}^{x_n} \int_{\phi(x)}^{f(x)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

Find the areas of the following surfaces: —

a) That cut off the *sphere*, $x^2 + y^2 + z^2 = a^2$ by the *cylinder*, $x^2 + y^2 = ax$.

Ans. $S = 4a^2.$

b) Of the solid in Ex. 8.

Ans. $S = 16a^2.$

c) The convex surface of the solid in Ex. 7.

Ans. $S = 4a^2 m.$

71. STATICAL MOMENT AND CENTRE OF GRAVITY.

A. Statical Moment.

Let the area $M_0 M_n P_n P_0$ in fig. 35 be one of the two equal faces of an indefinitely thin material solid, or lamina, of perfectly uniform thickness and density; *so that any two parts of it whose faces have equal areas will have exactly the same weight.*

Conceive this lamina to be rigidly attached to OX, and held fixedly in a horizontal position. Gravity will then act on each particle of the lamina in a direction perpendicular to the plane surface, $M_0 P_n$; and the sum of all the gravitational forces acting on these particles will yield a total force tending to turn the lamina about OX. This tendency to turn about OX is just balanced by the equal opposing force which holds the lamina in a horizontal position.

Any portion of the lamina, as $ABCD$, will exert a turning force about OX, which will depend upon (1) the area of its surface, (2) its distance from OX, (3) the thickness of the lamina, and (4) the density of the material of which the lamina is composed. The last two elements are, by hypothesis, constants throughout the lamina. Together, they determine the weight, m, per unit area of the lamina.

It follows from mechanical principles that the turning force exerted by the portion of the lamina whose surface is $ABCD$, will be measured by the product of the area of $ABCD$ by the distance MP, multiplied by the constant m.

Let us call this product **the statical moment of the area ABCD.** Denote it by ∂S_x and we have by definition,

$$(1) \qquad \partial S_x = m \times MP \times \textit{area } ABCD.$$

Let $y = f(x)$ be the equation of the curve $P_0 P_n$, and let the area $M_0 P_n$ be divided into parallel strips as explained in § 70. Let the coördinates of P be $OM = x$ and $MP = y$. Then the area of $ABCD = AB \times AD = dx\,dy$: and (1) gives,

$$(2) \qquad \partial S_x = m\,y\,dx\,dy.$$

This is the statical moment of $ABCD$ about OX.

In like manner *the statical moment of $ABCD$ about OY may be defined as,*

(3) $$\partial S_y = m\,x\,dx\,dy.$$

Now, the statical moments of the rectangular strip $M_k Q$ about OX and OY are, evidently, the following *partial* integrals:

(4) $$dS_x = \int_0^{y_k} m\,y\,dx\,dy = \frac{m}{2}\,y_k^2\,dx\,;$$

(5) $$dS_y = \int_0^{y_k} m\,x\,dx\,dy = m\,x\,y_k\,dx.$$

The limit of the sum of the statical moments of the rectangular strips $M_k Q$, etc., will obviously be the statical moment of the entire lamina, $M_0 M_n P_n P_0$. Integrating (4) and (5) between $OM_0 = a$ and $OM_n = b$ we obtain,*

(6) $$S_x = \int_a^b \int_0^y m\,y\,dx\,dy = \frac{m}{2}\int_a^b y^2 dx,$$

for the statical moment of the lamina about OX; and,

(7) $$S_y = \int_a^b \int_0^y m\,x\,dx\,dy = m\int_a^b x\,y\,dx,$$

for its statical moment about OY; where m is the weight per unit area of the lamina.

B. Centre of Gravity.

Expression (6) furnishes a value for the statical moment, S_x, of the entire area $M_0 P_n$, obtained by a double summation extending over the whole surface of the lamina; thus obtaining the total statical moment about OX by finding the limit of the sum of the statical moments of infinitesimal parts of the lamina.

Conceive, now, the total turning effect of gravity on the lamina to be replaced by a single force just equal to it, applied at such a point, K_g, that the turning effect, or statical moment about OX, of

* *The y_k in (4) and (5) is the ordinate of a point P_k on the curve $y = f(x)$. The point is not restricted in the integration, but varies from P_0 to P_n. Hence, in (6) and (7) we may replace the upper limit y_k by the general ordinate, y.*

the single force, is exactly equal to the total of the statical moments of the parts of the lamina, as found in (6).

The single force will be m *multiplied by the area of the lamina;* and, since this is a force of fixed amount and its statical moment is also fixed [being equal to S_x in (6)], the distance from OX of its point of application, K_g, is fixed. Call this distance $\bar{y}$ (*read it y-dash*). Then $\bar{y}$ is the ordinate of K_g. By the conditions explained we should have,

$$(8) \qquad S_x = \bar{y} \times m \times \textit{area of lamina}$$

$$= \bar{y} m \int_a^b \int_0^y dx\, dy = \bar{y} m \int_a^b y\, dx\,;$$

where $y = f(x)$ *is the equation of the arc* $P_0 P_n$. Setting the values of S_x from (6) and (8) equal to each other and solving for $\bar{y}$ we get,

$$(9) \qquad \bar{y} = \frac{\int_a^b \int_0^y y\, dx\, dy}{\int_a^b \int_0^y dx\, dy} = \frac{\int_a^b y^2 dx}{2\int_a^b y\, dx}.$$

This equation determines the ordinate, $\bar{y}$, of the point, K_g, but does not determine the point further than to limit it to the straight line, $y = \bar{y}$, parallel to OX.

But we may apply the same considerations to S_y in (7), and in like manner obtain the abscissa, $\bar{x}$, (*read it x-dash*) of the point, K_g. We should obtain,

$$(10) \qquad S_y = \bar{x} m \int_a^b \int_0^y dx\, dy = \bar{x} m \int_a^b y\, dx.$$

Whence, comparing with (7) and solving, we find,

$$(11) \qquad \bar{x} = \frac{\int_a^b \int_0^y x\, dx\, dy}{\int_a^b \int_0^y dx\, dy} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}.$$

This point, K_g $(\bar{x}, \bar{y})$, is now determined, and can be found for a given area under the curve, $y = f(x)$, whenever we can find the values of the integrals in (9) and (11).

It is called **The Centre of Gravity** *of the area.*

The student will observe that the m disappears in (9) and (11). Hence the position of K_g, relatively to OX and OY, is independent of both the thickness of the lamina and the material composing it; *provided the lamina is of perfectly uniform thickness, and any two equal areas have the same weight.*

From the mathematical point of view, it is of no consequence whether the student regards equations (9) and (11) as *arbitrarily defining* "*centre of gravity*," or whether he regards them as deduced from the preceding mechanical considerations. In any event, these considerations will inform him as to the mechanical interpretation of formulæ (9) and (11); and will outline the method by which he may solve other mechanical problems concerning the centre of gravity of a lamina, or plane area.

Other formulæ for the centre of gravity are given here for reference. *In each case the object is assumed to be of uniform density.*

If $r = f(\theta)$ is the polar equation of a curve, the centre of gravity of the sectorial area between the curve and two radii-vectores corresponding to $\theta = \theta_0$ and $\theta = \theta_n$ will be given by the formulæ,

$$(12)\quad \bar{x} = \frac{\int_{\theta_0}^{\theta_n}\int_0^r r^2 \cos\theta \, d\theta \, dr}{\int_{\theta_0}^{\theta_n}\int_0^r r \, d\theta \, dr} = \frac{2}{3}\,\frac{\int_{\theta_0}^{\theta_n} r^3 \cos\theta \, d\theta}{\int_{\theta_0}^{\theta_n} r^2 \, d\theta};$$

$$(13)\quad \bar{y} = \frac{\int_{\theta_0}^{\theta_n}\int_0^r r^2 \sin\theta \, d\theta \, dr}{\int_{\theta_0}^{\theta_n}\int_0^r r \, d\theta \, dr} = \frac{2}{3}\,\frac{\int_{\theta_0}^{\theta_n} r^3 \sin\theta \, d\theta}{\int_{\theta_0}^{\theta_n} r^2 \, d\theta}.$$

If the curve $y = f(x)$ is rotated about OX, the centre of gravity of the **volume** *generated by the area under the curve from $x = a$ to $x = b$ will be given by $\bar{y} = 0$, and,*

$$(14)\quad \bar{x} = \frac{\int_a^b\int_0^y xy \, dx \, dy}{\int_a^b\int_0^y y \, dx \, dy} = \frac{\int_a^b xy^2 \, dx}{\int_a^b y^2 \, dx};$$

and the centre of gravity of the **surface** *generated by the arc of* $y = f(x)$ *between* $x = a$ *and* $x = b$, *when the arc is rotated about* OX, *is the point* $\bar{y} = 0$ *and,*

$$(15)\quad \bar{x} = \frac{\int_a^b xy\sqrt{dx^2 + dy^2}}{\int_a^b y\sqrt{dx^2 + dy^2}}.$$

The centre of gravity of a fine wire bent into the form of an arc of the curve, $y = f(x)$, *will be given by the formulæ,*

$$(16)\quad \bar{x} = \frac{\int_a^b x\sqrt{dx^2 + dy^2}}{\int_a^b \sqrt{dx^2 + dy^2}},\quad \bar{y} = \frac{\int_a^b y\sqrt{dx^2 + dy^2}}{\int_a^b \sqrt{dx^2 + dy^2}}.$$

For other formulæ, and for demonstrations of these here given, the student should consult a good treatise on analytic mechanics.

Exercises.

Find the centre of gravity of the following:—

1. Of the area of *the triangle*, $y = mx$, $y = m_1x$, $x = h$.

Ans. $\bar{x} = \dfrac{2h}{3}$.

2. Of the area of *the circular sector*, $r = a$; (1) from $\theta = -\alpha$ to $\theta = \alpha$; (2) $\alpha = \dfrac{\pi}{4}$; and (3) $\alpha = \dfrac{\pi}{2}$.

Ans. (1) $\bar{x} = \dfrac{2a\sin\alpha}{3\alpha}$; (2) $\bar{x} = \dfrac{4a\sqrt{2}}{3\pi}$; (3) $\bar{x} = \dfrac{4a}{3\pi}$.

3. Of the area under one arch of *the cycloid*,

$$x = a(\phi - \sin\phi),\quad y = a(1 - \cos\phi).$$

Ans. $\bar{x} = \pi a$, $\bar{y} = \dfrac{5a}{6}$.

4. Of the area of the first quadrant of *the ellipse*, $b^2x^2 + a^2y^2 = a^2b^2$.

Ans. $\bar{x} = \dfrac{4a}{3\pi}$, $\bar{y} = \dfrac{4b}{3\pi}$.

5. Of the area of the first quadrant of *the astroid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $\bar{x} = \bar{y} = \dfrac{256a}{315\pi}$.

6. Of the area of *the cardioid*, $r = a(1 + \cos\theta)$.

Ans. $\bar{x} = \frac{5a}{6}$, $\bar{y} = 0$.

7. Of the area of a loop of the curve, $r = a\cos 2\theta$.

Ans. $\bar{x} = \frac{128a\sqrt{2}}{105\pi}$, $\bar{y} = 0$.

8. Of the volume of *the cone* whose altitude is h, generated by rotating the line, $y = mx$, about OX.

Ans. $\bar{x} = \frac{3h}{4}$.

9. Of the volume of *the spherical sector* generated by the circular sector cut off the circle, $x^2 + y^2 = a^2$, by the lines, $y = x\tan\alpha$ and $y = 0$.

Ans. $\bar{x} = \frac{3}{8}a(1 + \cos\alpha)$.

10. Of the volume of *the paraboloid* whose altitude is h, generated by rotating the parabola, $y^2 = 4px$, about OX.

Ans. $\bar{x} = \frac{2h}{3}$.

11. Of the area of *the spherical zone* cut off by the planes $x = b$ and $x = c$ from the sphere generated by $x^2 + y^2 = a^2$.

Ans. $\bar{x} = \frac{1}{2}(b + c)$.

12. Of *the semi-circumference* of $x^2 + y^2 = a^2$ above the x-axis.

Ans. $\bar{x} = 0$, $\bar{y} = \frac{2a}{\pi}$.

13. Of *the arc* of one arch of *the cycloid*, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

Ans. $\bar{x} = \pi a$, $\bar{y} = \frac{4a}{3}$.

14. Of *the arc* of *the astroid*, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, in the first quadrant.

Ans. $\bar{x} = \bar{y} = \frac{2a}{5}$.

72. MOMENT OF INERTIA AND RADIUS OF GYRATION.

A. Moment of Inertia.

The **Moment of Inertia** *of an infinitesimal body, or particle, with respect to an axis is the product of its mass* by the square of its distance from the axis. It may be denoted by* **I**.

---

* The mass of any body is its weight divided by the gravitation constant, g. *Mass is jointly proportional to volume and density.*

The moment of inertia of an arc (a material curve or line) or of an area (an indefinitely thin lamina) or of a material solid, with respect to an axis, is the sum of the moments of all its particles with respect to that axis.

For example, the moment of inertia of the homogeneous lamina M_0P_n in fig. 35 with respect to OX can be found as follows: —

The mass of the part $ABCD$ is equal to the product of the area $AB \times AD$, $= dx\,dy$, by the mass of unit area, which may be denoted by m. Let MP, $= y$, be the distance of the infinitesimal element, $ABCD$, from OX. Denote the moment of inertia of $ABCD$ with respect to OX by ∂I_x. Then we have by definition,

$$(1) \qquad \partial I_x = (MP)^2 m\,dx\,dy = my^2\,dx\,dy.$$

The moment of inertia of the rectangular strip M_kQ about OX will be the limit of the sum of the moments of its parts, — of which $my^2\,dx\,dy$ is one. This may be obtained by the summation in which y varies from 0 to y_k by the equal infinitesimal increments dy, $= AD$, and in which dx, $= AB$, is a constant. This may be obtained by integrating $my^2\,dx\,dy$ from 0 to y_k. In short, *if we denote the moment of inertia of the rectangular strip, M_kQ,* by dI_x *we get,*

$$(2) \qquad dI_x = m\int_0^{y_k} y^2\,dx\,dy = \frac{m}{3}\,y_k^3\,dx.$$

The moment of inertia of the lamina, M_0P_n, about OX will be the limit of the sum of the moments of inertia of such rectangular strips as M_kQ; that is, will be the limit of the sum of the terms dI_x given by (2) when x varies from a to b.

Denote the moment of inertia of the lamina M_0P_n about OX by I_x and we have,

$$(3) \qquad I_x = m\int_a^b\int_0^y y^2\,dx\,dy = \frac{m}{3}\int_a^b y^3\,dx.$$

This is a general formula for the moment of inertia about OX of the area (or lamina) under a curve, $y = f(x)$, and between the parallel lines $x = a$ and $x = b$; provided the lamina is of perfectly uniform thickness and density.

In a similar manner, *if we denote by I_y the moment of inertia of the lamina, $M_0 P_n$, about OY* we shall obtain,

$$(4) \qquad I_y = m\int_a^b \int_0^y x^2 dx\, dy = m\int_a^b x^2 y\, dx.$$

If an axis is drawn through O perpendicular to the plane XOY the moment of inertia of the lamina, $M_0 P_n$, with respect to this axis may be shown to be,

$$(5) \qquad I_p = m\int_a^b \int_0^y (x^2 + y^2)\, dx\, dy = m\int_a^b (x^2 y + \tfrac{1}{3} y^3)\, dx.$$

This is called **the polar moment of inertia** *of the lamina.*

If a homogeneous solid is bounded by the surface, $z = f(x, y)$ [rectangular space axes, OX, OY, and OZ being used] the solid may be cut up into infinitesimal elements by three sets of parallel planes perpendicular respectively to the three axes, and at distances dx, dy, and dz apart. An infinitesimal element of the mass of the solid will be $m\, dx\, dy\, dz$, where m is the mass of unit volume: and the moment of inertia of the entire solid about each of the three axes can be shown to be,

$$(6) \qquad I_z = \int\int\int m\,(x^2 + y^2)\, dx\, dy\, dz,$$

$$(7) \qquad I_y = \int\int\int m\,(x^2 + z^2)\, dx\, dy\, dz,$$

$$(8) \qquad I_x = \int\int\int m\,(y^2 + z^2)\, dx\, dy\, dz.$$

The limits of each of the three integrations in each of the three formulæ must be supplied by a study of the given surface. For example, if it is the sphere, $x^2 + y^2 + z^2 = r^2$, we get,

$$(9) \qquad I_z = 8m\int_0^r \int_0^{\sqrt{r^2 - x^2}} \int_0^{\sqrt{r^2 - x^2 - y^2}} (x^2 + y^2)\, dx\, dy\, dz.$$

Exercise.

Find I_z from equation (9).

The student will have little difficulty in adapting formulæ for use when the moment of inertia of a curve is sought. *In this case the infinitesimal element of mass is,*

$$(10) \qquad m\,ds = m\,\sqrt{dx^2 + dy^2},$$

for a plane curve $y = f(x)$; *and,*

$$(11) \qquad m\,ds = m\,\sqrt{dx^2 + dy^2 + dz^2},$$

when it is a space curve, and m *is mass of unit length.*

B. Radius of Gyration.

Denote by M the entire mass of a given body (solid, or lamina, or curve). Let I represent its moment of inertia with respect to a given axis. Let k be such a number that,

$$(12) \qquad Mk^2 = I\,, \quad \text{or} \quad k = \sqrt{I \div M}.$$

Then k *is called* **the Radius of Gyration** *with respect to the given axis.*

If the body is a line or curve, $M = m \times$ *its length*; if it is a lamina, $M = m \times$ *area of lamina*; and if the body is a solid, $M = m \times$ *volume of solid.*

In each case $m =$ *mass* of unit length, or of unit area, or of unit volume, according to the body treated.

If a body is conceived as rotating about the axis with respect to which its moment of inertia is taken, it will behave, dynamically, as if its whole mass were concentrated into a thin circular ring whose radius is k, $=$ radius of gyration, and whose axis is coincident with the axis of rotation.

Example.

Find the moment of inertia, and the radius of gyration, of a circular lamina of radius a about a tangent to the circle.

Solution. The equation of a circle referred to a tangent and diameter as axes is, $x^2 + y^2 - 2ax = 0$, when $a =$ radius and the y-axis is the tangent. Taking moment of inertia about OY we get,

$$(13) \qquad I_y = 2m\int_0^{2a}\int_0^{y} x^2\,dx\,dy = 2m\int_0^{2a} x^2\sqrt{2ax - x^2}\,dx$$

$$= 2a^4 m\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\phi + 2\sin\phi\cos^2\phi + \sin^2\phi\cos^2\phi)\,d\phi = \frac{5}{4}\,\pi a^4 m.$$

[*The radical* $\sqrt{2ax - x^2}$, $\equiv \sqrt{a^2 - (x-a)^2}$, *is rationalized by putting* $x - a = a \sin\phi$.]

The mass, M, of the circle is $m \times area = \pi a^2 m$. Substituting in (12) we get,

$$k_y = \sqrt{\frac{5}{4}\,\pi a^4 m \div \pi a^2 m} = \frac{a}{2}\sqrt{5},$$

which is the radius of gyration of the circle of radius a about one of its tangents.

Exercises.

Find the moment of inertia and the radius of gyration of the following:—

1. Of *the rectangular lamina* whose edges are a and b about an axis (1) through a vertex, and normal to its plane; (2) through the intersection of its diagonals, and normal to its plane.

Ans. (1) $I = \frac{M}{3}(a^2 + b^2)$, $k = \frac{1}{3}\sqrt{3(a^2+b^2)}$;

(2) $I = \frac{M}{12}(a^2 + b^2)$, $k = \frac{1}{12}\sqrt{12(a^2+b^2)}$.

*2. Of *the rectangular lamina* whose edges are $2a$ and $2b$ about an axis normal to its plane, and at a distance c from the intersection of its diagonals.

Ans. $I = \frac{M}{3}(a^2 + b^2 + 3c^2)$.

3. Of *the circular lamina* whose radius is a about (1) a diameter; (2) an axis through its centre, and normal to its plane.

Ans. (1) $I = M\frac{a^2}{4}$, $k = \frac{a}{2}$; (2) $I = M\frac{a^2}{2}$, $k = \frac{a\sqrt{2}}{2}$.

4. Of *an elliptical lamina* whose semi-axes are a and b about (1) its major axis; (2) a normal to its plane through the centre.

Ans. (1) $I = M\frac{b^2}{4}$, $k = \frac{b}{2}$; (2) $I = M\frac{a^2+b^2}{4}$, $k = \frac{1}{2}\sqrt{a^2+b^2}$.

5. Of *a straight line* whose length is a about an axis through one end, and normal to it.

Ans. $I = \frac{1}{3}Ma^2$; $k = \frac{a}{3}\sqrt{3}$.

6. Of *an arc of the circle* whose radius is a, and whose angle at the centre is 2α, (1) about the normal to the plane of the circle through the centre; (2) about an axis through the middle point of the arc, and normal to its plane.

Ans. (1) $I = Ma^2$, $k = a$; (2) $I = 2Ma^2\left(1 - \frac{\sin\alpha}{\alpha}\right)$.

7. The I and k of a solid of rotation may be found by dividing it into thin circular laminæ perpendicular to its axis. If $y = f(x)$ is the generatrix, and the rotation is about the x-axis, then $\pi m y^2 dx$ is the mass of one of the circular laminæ, and (by Ex. 3) its radius of gyration is $k_1 = \frac{y\sqrt{2}}{2}$. Hence, the I_x of one of the laminæ is $k_1^2 \times mass = \frac{y^2}{2} \times \pi m y^2 dx$: and the I_x and the k_x of the part of the solid included between two parallel planes $x = a$ and $x = b$ are,

$$I_x = \frac{\pi m}{2}\int_a^b y^4\,dx, \quad \text{and } k_x^2 = \frac{1}{2}\int_a^b y^4\,dx \div \int_a^b y^2\,dx.$$

8. Find I and k about the x-axis for the cylinder generated by rotating about OX the area under the line $y = c$, from $x = a$ to $x = b$.

Ans. $I = \frac{1}{2} Mc^2$; $k^2 = \frac{c^2}{2}$.

9. Find I and k about the x-axis for the sphere generated by $x^2 + y^2 = a^2$.

Ans. $I = \frac{2}{5} Ma^2$; $k^2 = \frac{2a^2}{5}$.

10. Find I and k about the x-axis for the right circular cone generated by the area under the line $y = \frac{a}{h}x$, from $x = 0$ to $x = h$.

Ans. $I = \frac{3}{10} Ma^2$; $k^2 = \frac{3a^2}{10}$.

11. Devise a method and find k about the x-axis for the *spherical shell* generated by the circumference of the circle, $x^2 + y^2 = a^2$.

Ans. $k^2 = \frac{1}{4\pi a^2}\int_{-a}^{a} y^2 \times 2\pi a\,dx = \frac{2a^2}{3}$.

12. Find I and k about the x-axis for *the elliptical cylinder* whose semi-axes are a and b, and whose altitude is h.

Ans. $I = M\frac{a^2 + b^2}{4}$, $k = \frac{1}{2}\sqrt{a^2 + b^2}$.

13. Find I and k about the x-axis for *the ellipsoid*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans. $I = \frac{M}{5}(b^2 + c^2)$; $k = \frac{1}{5}\sqrt{5(b^2 + c^2)}$.

CHAPTER XXXII.

VELOCITIES, RATES, AND ACCELERATION.

73. VELOCITIES, RATES, AND ACCELERATION.

A. Uniform Velocity.

Let a point, P, be in steady motion along any continuous path (*straight or curved*); so that in all equal time-intervals, however chosen, P will traverse equal space-, or distance-, intervals. Then the space, or distance, traversed in unit length of time, is the same everywhere throughout the motion; and *the motion is called* **uniform.**

In uniform motion *the distance traversed in unit time is called* **the velocity of uniform motion.** *It is constant.* It is called, also, **rate of motion.**

If we denote the velocity of uniform motion by v, then the space, s, traversed in time, t, will obviously be given by the equation,

(1) $$s = vt, \quad \text{whence } v = s \div t.$$

This equation being true for any time, t, whatever, *provided* the space traversed in that time is s, it follows that: —

The velocity of uniform motion may be obtained by dividing the length of any part of the path whatsoever by the time consumed in traversing it.

By *the velocity of P at any point P'* of its path, is meant: *the space which P will traverse in the first unit of time after passing P'.* In uniform motion the velocity at all points is the same, or is constant, and does not depend upon either the position of P' or upon elapsed time; hence, *v is not a function of t.*

B. Variable Velocity.

Let the motion of P along any continuous path (*straight or curved*) be such that in any series (*however chosen*) of successive, equal

time-intervals the successive, corresponding space-intervals traversed are *unequal.* These space-intervals may be increasing, or decreasing; or may now increase, then decrease, or *vice versa.* To fix the ideas, let the spaces be assumed to be all the while *increasing;* that is, let the speed, or rate of motion, be increasing.

In this case it is obvious that if s' denote the distance of P' (*which is any position of* P) from the fixed point, A, of its path (see fig. 36), and if t' denote the time (*measured in terms of some convenient time-unit*) consumed in traversing s', then the number of space units traversed by P in unit-time will continually increase, as P moves towards B; and "velocity" as defined for uniform motion is wholly indeterminate. There will be as many "velocities" as there are positions of P between its first and last positions. In short, the "velocity" of P is a variable, and is a function of time reckoned from the beginning of the motion.

A measure of **The velocity of variable motion at a given point or instant,** may be obtained as follows: —

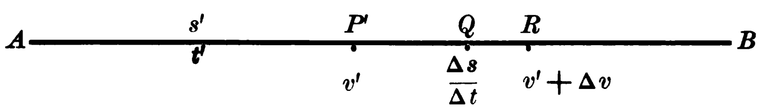

Fig. 36.

Let P' be the position of P at end of time t' after passing A. It is required to find the velocity, v', of P at P'; that is, *How far would P move in unit time after passing P' if the motion instantaneously became uniform at P', and remained so during unit time?*

Give t' the increment Δt; and let $\Delta s, = P'R$, be the space traversed by P in time Δt. Since the speed is, by hypothesis, increasing, the velocity of P is greater at R than it was at P'. Denote the velocity of P at R by $v' + \Delta v$; then Δv is positive and $v' + \Delta v > v'$.

The ratio $\Delta s \div \Delta t$ is obviously the velocity with which P would have to move in order to traverse the space $\Delta s, = P'R$, in time Δt; *provided* the motion were uniform. Call this *the mean, or average, velocity* of P during time Δt after passing P'.

The mean velocity of P over space $P'R$, is, obviously, greater than the velocity of P at P', and less than that at R; and must be the actual velocity of P at some point, Q, between P' and R.

Now make $\Delta t \doteq 0$; then $\Delta s, = P'R$, will approach zero, and the points Q and R will approach P'. Hence, the ratio $\Delta s \div \Delta t, =$ velocity of P at Q, will have for its limit, when Q approaches P', the velocity, v', of P at P'.

Expressing this formally we have,

$$(2) \quad v' < \frac{\Delta s}{\Delta t} < v' + \Delta v, \text{ whence } v' = \lim_{\Delta t \doteq 0}\left[\frac{\Delta s}{\Delta t}\right] = D_t s = \frac{ds}{dt}.$$

Hence, the velocity of a variable motion at a given point P' of the path (or at a given instant) of the motion may be found by differentiating the space, s, (expressed as a function of time, t,) with respect to t, and substituting in this derivative the values of s and t at the point considered.

Example.

A body falling freely from rest (in a vacuum) will traverse, in t seconds, a space in feet of $s = \frac{1}{2} g t^2$. Hence, its velocity v at the end of t seconds will be,

$$(3) \qquad v = \frac{ds}{dt} = gt. \qquad (g = 32.2, \textit{ approximately.})$$

The relation between s and t for any given problem in motion must be expressed in the form of an equation $\phi(s, t) = 0$, or $s = f(t)$. *This is called the equation of the motion.*

If $s, = f(t)$, is given we may find $v, = \frac{ds}{dt}$, by differentiation.

Conversely, if $v, = \frac{ds}{dt} = f'(t)$, is given we may find,

$$(3) \qquad s = \int ds = \int v\,dt = \int f'(t)\,dt = f(t) + k,$$

by anti-differentiation.

If the velocity were decreasing, instead of increasing, the case could be treated in precisely the same manner as in the foregoing; and with the same result for v.

If the velocity changes sign the direction of the motion is reversed.

C. Rates.

The applications of mathematics furnish many examples of continuously varying magnitudes which require time to undergo any specified amount of change in value; that is, of variable quantities which are functions of time.

Let $\zeta = f(t)$ be any such function. The successive values of ζ may be geometrically represented by the successive distances of a point P from a fixed point A on a straight line; and the motion of P on the line will represent perfectly the variation of ζ as a function of t. *The velocity of P, or the rate of motion of P, will then represent exactly* **the rate of change in value of ζ.**

Fig. 37.

It follows, therefore, from § 73, *B*, that *the derivative with respect to t, of any magnitude, ζ, which is a function of time, t, furnishes a measure of the rate of change of value of ζ.*

If $\dfrac{d\zeta}{dt}$ is positive, ζ is increasing; and if $\dfrac{d\zeta}{dt}$ is negative, ζ is decreasing. (*See* § 15.)

D. Acceleration.

The motion of a point P is said to be *accelerated* when its velocity is a function of time, or is variable. If the velocity is increasing its motion is positively accelerated; and if its velocity is decreasing its motion is negatively accelerated, or retarded.

By **Acceleration** *is meant, therefore, the rate of change of velocity.*

Denoting acceleration by α, velocity by v, and space by s, $= f(t)$, we have,

$$(4) \qquad \alpha = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \text{ since } v = \frac{ds}{dt}.$$

Acceleration is said to be *uniform* when $\alpha = a$ *constant.* For example, if $s = \frac{1}{2}gt^2$, $v = gt$, and $\alpha = g$. This means that a body falling freely in a vacuum, will have g feet added to its velocity during each second of its fall. (See example, § 73, *B*.)

If $\alpha = 0$ the velocity is a constant and the motion is uniform.

Exercises.

1. If $s = a(e^t + e^{-t})$ find v and a.

2. If $a = g = a$ *constant*, show that the general equation of uniformly accelerated motion is $s = \frac{1}{2}gt^2 + k_1 t + k_0$.

Show, also, that $k_1 =$ *initial velocity of the motion* (*when* $t = 0$); and $k_0 =$ *space already traversed when* $t = 0$; and hence, if a body, moving uniformly, starts from rest, and time is reckoned from the instant of starting, the most general equation of its motion is $s = \frac{1}{2}gt^2$.

3. Show that if a body is thrown vertically upward with an initial velocity v_0, then its equation of motion is, $s = v_0 t - \frac{1}{2}gt^2$.

4. Find from (3) formulæ for the height reached, and the time of ascent, of a ball shot vertically upward.

If a ball is shot vertically upward with an initial velocity of 1200 per second, how high will it rise and what is the time of its ascent?

5. The acceleration of a moving point varies directly with the time; find the most general form of its law of motion.

Find, also, the most general form of its law of motion if the acceleration varies inversely with the time.

6. Find the law of motion when the acceleration is $a = \sin t$: also, when the velocity varies inversely with the square root of the time.

7. The radius of a sphere is increasing at the rate of one inch per second. How fast is the surface changing? — how fast the volume? — in each case when $r = 10$ inches.

8. Let P move on a plane curve referred to rectangular axes OX and OY. Denote by v, v_x, and v_y the velocities of P (1) along the curve, (2) parallel to OX and (3) parallel to OY. Show that (*a*) $v_x = \dfrac{dx}{dt}$, (*b*) $v_y = \dfrac{dy}{dt}$ and (*c*) $v = \sqrt{v_x^2 + v_y^2}$, at the point (x, y) on the curve.

Show, also, that if $\tau =$ angle between the tangent at (x, y) and OX, then $v_x = v\cos\tau$ and $v_y = v\sin\tau$.

[SUGGESTION: *Use* (4) § 64 and (8), (9) § 65.]

9. If $P(x, y, z)$ is a point on a space curve referred to rectangular axes, OX, OY, OZ, and v_x, v_y, v_z, and v denote the velocities of P parallel to the x —, y —, and z-axes, and along the curve, show that,

$$v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt},\ \text{and}\ v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Show, also, that if α, β, and γ are the direction angles of the curve at P then,

$$v_x = v\cos\alpha,\ v_y = v\cos\beta,\ \text{and}\ v_z = v\cos\gamma.$$

[SUGGESTION: *Use* ds, *etc. of exs.* 9, 10, p. 198.]

10. Show that, in polar plane coördinates when $P(r, \theta)$ is a moving point on the curve $r = f(\theta)$ and $a =$ *the angle* between OP and the tangent at P:

a) *the velocity of P along the curve is* $v = \dfrac{ds}{dt} = \sqrt{\left(\dfrac{dr}{dt}\right)^2 + r^2\left(\dfrac{d\theta}{dt}\right)^2}$.

b) *the velocity of P along r, $= OP$, is* $v_r = \dfrac{dr}{dt}$;

c) the velocity of P normal to the radius vector OP, is $v_\theta = v \sin a = r\dfrac{d\theta}{dt}$;

d) and $v = \sqrt{v_r^2 + v_\theta^2}$.

[SUGGESTION: See §§ 62*a*, 62*b*.]

11. The path of a projectile fired at an elevation a, and with an initial velocity v_0, is given by the equation,

$$y = x \tan a - \frac{g x^2}{2 v_0^2 \cos^2 a}.$$

Given $v_x = v_0 \cos a$, find v_y and v. Find, also, the *acceleration*, a_y, of v_y.

Ans. $v_y = v_0 \sin a - \dfrac{g x}{v_0 \cos a}$, $v = \sqrt{v_0^2 - 2 g y}$, $a_y = - g$.

APPENDIX.

FORMULÆ OF DIFFERENTIATION AND ANTI-DIFFERENTIATION.

I. Differential Formulæ.

1. $dk = 0.$ 2. $d(ku) = k\,du.$ 3. $d(u \pm v) = du \pm dv.$

4. $d(uv) = v\,du + u\,dv.$ 5. $d(u \div v) = [v\,du - u\,dv] \div v^2.$

6. $d[f(u) + k] = df(u) = D_u f(u)\,du = f'(u)\,du.$

7. $du^n = nu^{n-1}du.$ 8. $d\sqrt{u} = du \div 2\sqrt{u}.$

9. $d\log_a u = (\log_a e\,du) \div u.$ 10. $da^u = a^u \log_e a\,du.$

11. $d\log_e u = du \div u.$ 12. $de^u = e^u du.$

13. $d\sin u = \cos u\,du.$ 14. $d\sin^{-1}u = du \div \sqrt{1-u^2}.$

15. $d\cos u = -\sin u\,du.$ 16. $d\cos^{-1}u = -du \div \sqrt{1-u^2}.$

17. $d\tan u = \sec^2 u\,du.$ 18. $d\tan^{-1}u = du \div (1+u^2).$

19. $d\,\mathrm{ctn}\,u = -\csc^2 u\,du.$ 20. $d\,\mathrm{ctn}^{-1}u = -du \div (1+u^2).$

21. $d\sec u = \sec u \tan u\,du.$ 22. $d\sec^{-1}u = du \div u\sqrt{u^2-1}.$

23. $d\csc u = -\csc u\,\mathrm{ctn}\,u\,du.$ 24. $d\csc^{-1}u = -du \div u\sqrt{u^2-1}.$

25. $d\,\mathrm{vers}\,u = \sin u\,du.$ 26. $d\,\mathrm{vers}^{-1}u = du \div \sqrt{2u-u^2}.$

27. $d\,\mathrm{covers}\,u = -\cos u\,du.$ 28. $d\,\mathrm{covers}^{-1}u = -du \div \sqrt{2u-u^2}.$

29. $\partial_x z = \dfrac{\partial z}{\partial x}dx,\ \partial_y z = \dfrac{\partial z}{\partial y}dy.$ 30. $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy.$

31. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz.$

II. Anti-differential Formulæ.

32. $\int df(u) = \int f'(u)\,du = f(u) + k.$ 33. $d\int f(u)\,du = f(u)\,du.$

34. $\int kf(u)\,du = k\int f(u)\,du.$ 35. $\int [U\,du \pm V\,du] = \int U\,du \pm \int V\,du.$

36. $\int u\,dv = uv - \int v\,du.$

(a) *Trigonometric Forms.*

37. $\int \sin u\,du = -\cos u.$

38. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2}.$

39. $\int \cos u\,du = \sin u.$

40. $\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2}.$

41. $\int \tan u\,du = \log\sec u.$

42. $\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\log(1+u^2).$

43. $\int \operatorname{ctn} u\,du = \log\sin u.$

44. $\int \operatorname{ctn}^{-1}u\,du = u\operatorname{ctn}^{-1}u + \frac{1}{2}\log(1+u^2).$

45. $\int \sec u\,du = \log\tan\left(\frac{\pi}{4}+\frac{u}{2}\right).$

46. $\int \operatorname{vers} u\,du = u - \sin u.$

47. $\int \csc u\,du = \log\tan\frac{u}{2}.$

48. $\int \operatorname{covers} u\,du = u + \cos u.$

49. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{2}\sin u\cos u.$

50. $\int \tan^2 u\,du = \tan u - u.$

51. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{2}\sin u\cos u.$

52. $\int \operatorname{ctn}^2 u\,du = -\operatorname{ctn} u - u.$

53. $\int \sin^n u\,du = \frac{\sin^{n-1}u\cos u}{-n} + \frac{n-1}{n}\int \sin^{n-2}u\,du.$

54. $\int \sin^n u\,du = \frac{\sin^{n+1}u\cos u}{n+1} + \frac{n+2}{n+1}\int \sin^{n+2}u\,du.$

55. $\int \cos^n u\,du = \frac{\sin u\cos^{n-1}u}{n} + \frac{n-1}{n}\int \cos^{n-2}u\,du.$

56. $\int \cos^n u\,du = -\frac{\sin u\cos^{n+1}u}{n+1} + \frac{n+2}{n+1}\int \cos^{n+2}u\,du.$

57. $\int \tan^n u\,du = \frac{\tan^{n-1}u}{n-1} - \int \tan^{n-2}u\,du.$

58. $\int \operatorname{ctn}^n u\,du = -\frac{\operatorname{ctn}^{n-1}u}{n-1} - \int \operatorname{ctn}^{n-2}u\,du.$

59. $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du.$

60. $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du.$

61. $\int \sin^m u\cos^n u\,du = \frac{\sin^{m+1}u\cos^{n-1}u}{m+n} + \frac{n-1}{m+n}\int \sin^m u\cos^{n-2}u\,du.$

62. $\int \sin^m u\cos^n u\,du = -\frac{\sin^{m-1}u\cos^{n+1}u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}u\cos^n u\,du.$

63. $\int \sin mu\sin nu\,du = \frac{\sin(m-n)u}{2(m-n)} - \frac{\sin(m+n)u}{2(m+n)}.$

64. $\int \sin mu\cos nu\,du = -\frac{\cos(m-n)u}{2(m-n)} - \frac{\cos(m+n)u}{2(m+n)}.$

65. $\int \cos mu \cos nu\,du = \frac{\sin(m-n)u}{2(m-n)} + \frac{\sin(m+n)u}{2(m+n)}.$

66. $\int \frac{du}{a+b\cos u} = \frac{-1}{\sqrt{a^2-b^2}} \sin^{-1}\left[\frac{b+a\cos u}{a+b\cos u}\right]$, if $a > b$.

67. $\int \frac{du}{a+b\cos u} = \frac{1}{\sqrt{b^2-a^2}} \log\left[\frac{b+a\cos u+\sqrt{b^2-a^2}\sin u}{a+b\cos u}\right]$, if $a < b$.

(b) *Rational Algebraic Forms.*

68. $\int u^n du = \frac{u^{n+1}}{n+1}, n \gtrless -1.$ 69. $\int \frac{du}{u} = \log u.$ 70. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}.$

71. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\log\frac{a+u}{a-u}.$ 72. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\log\frac{u-a}{u+a}.$

73. $\int \frac{du}{(u-a)(u-b)} = \frac{1}{a-b}\log\frac{u-a}{u-b}.$

(c) *Irrational Algebraic Forms.*

74. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\log\left[\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right]$, $a > 0$.

75. $\int \frac{du}{u\sqrt{bu-a}} = \frac{2}{\sqrt{a}}\tan^{-1}\sqrt{\frac{bu-a}{a}}$, $a > 0$.

76. $\int \frac{du}{\sqrt{u}} = 2\sqrt{u}.$ 77. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}.$ 78. $\int \frac{u\,du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2}.$

79. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$

80. $\int u\sqrt{a^2-u^2}\,du = \frac{-1}{3}(a^2-u^2)^{\frac{3}{2}}.$

81. $\int \frac{du}{u\sqrt{a^2\pm u^2}} = \frac{1}{a}\log\left[\frac{u}{a+\sqrt{a^2\pm u^2}}\right].$ 82. $\int \frac{du}{\sqrt{(a^2-u^2)^3}} = \frac{u}{a^2\sqrt{a^2-u^2}}.$

83. $\int \frac{u^n du}{\sqrt{a^2-u^2}} = \frac{u^{n-1}\sqrt{a^2-u^2}}{-n} + \frac{a^2(n-1)}{n}\int \frac{u^{n-2}du}{\sqrt{a^2-u^2}}.$

84. $\int \frac{du}{\sqrt{u^2\pm a^2}} = \log(u+\sqrt{u^2\pm a^2}).$ 85. $\int \frac{u\,du}{\sqrt{u^2-a^2}} = \sqrt{u^2-a^2}.$

86. $\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\frac{u}{a}.$ 87. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}.$

88. $\int \sqrt{u^2\pm a^2}\,du = \frac{u}{2}\sqrt{u^2\pm a^2} \pm \frac{a^2}{2}\log(u+\sqrt{u^2\pm a^2}).$

89. $\int \frac{u\,du}{\sqrt{a^2+u^2}} = \sqrt{a^2+u^2}.$ 90. $\int u\sqrt{a^2+u^2}\,du = \frac{1}{3}(a^2+u^2)^{\frac{3}{2}}.$

91. $\int u^n\sqrt{a^2+u^2}\,du = \frac{u^{n+1}\sqrt{a^2+u^2}}{n+2} + \frac{a^2}{n+2}\int\frac{u^n\,du}{\sqrt{a^2+u^2}}.$

92. $\int\frac{u^n\,du}{\sqrt{a^2+u^2}} = \frac{u^{n-1}\sqrt{a^2+u^2}}{n} - \frac{a^2(n-1)}{n}\int\frac{u^{n-2}\,du}{\sqrt{a^2+u^2}}.$

93. $\int\frac{du}{\sqrt{2au-u^2}} = \text{vers}^{-1}\frac{u}{a}.$ 94. $\int\frac{du}{u\sqrt{2au-u^2}} = \frac{\sqrt{2au-u^2}}{-au}.$

95. $\int\sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\text{vers}^{-1}\frac{u}{a}.$

96. $\int\frac{du}{(u+1)\sqrt{u^2-1}} = \sqrt{\frac{u-1}{u+1}}.$ 97. $\int\frac{du}{(u-1)\sqrt{u^2-1}} = -\sqrt{\frac{u+1}{u-1}}.$

98. $\int\sqrt{\frac{1+u}{1-u}}\,du = \sin^{-1}u - \sqrt{1-u^2}.$

99. $\int\frac{du}{\sqrt{(u-a)(b-u)}} = 2\sin^{-1}\sqrt{\frac{u-a}{b-a}},\ b>a.$

(*d*) *Exponential Forms.*

100. $\int a^u\,du = \frac{a^u}{\log a}.$ 101. $\int e^u\,du = e^u.$ 102. $\int ue^{au}\,du = \frac{e^{au}}{a^2}(au-1).$

103. $\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du.$

104. $\int\frac{e^{au}\,du}{u^n} = \frac{-e^{au}}{(n-1)u^{n-1}} + \frac{a}{n-1}\int\frac{e^{au}\,du}{u^{n-1}}.$

105. $\int e^{mu}\sin nu\,du = \frac{e^{mu}(m\sin nu - n\cos nu)}{m^2+n^2}.$

106. $\int e^{mu}\cos nu\,du = \frac{e^{mu}(m\cos nu + n\sin nu)}{m^2+n^2}.$

III. Integrals.

107. $\int_0^{\frac{\pi}{2}}\sin^n u\,du = \int_0^{\frac{\pi}{2}}\cos^n u\,du = \frac{1\cdot 3\cdot 5\cdot\ldots\ldots\cdot(n-1)}{2\cdot 4\cdot 6\cdot\ldots\ldots\ldots\cdot n}\,\frac{\pi}{2},$ n even integer.

108. " " $= \frac{2\cdot 4\cdot 6\cdot\ldots\ldots\cdot(n-1)}{1\cdot 3\cdot 5\cdot\ldots\ldots\ldots\cdot n},$ n odd integer.

109. $$\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2 u}\,du = \frac{\pi}{2}\Big[1-\Big(\frac{1}{2}\Big)^2k^2-\Big(\frac{1\cdot 3}{2\cdot 4}\Big)^2\frac{k^4}{3} - \Big(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Big)^2\frac{k^6}{5} - \ldots\ldots\Big],\ k^2<1.$$

110. $$\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-k^2\sin^2 u}} = \frac{\pi}{2}\Big[1+\Big(\frac{1}{2}\Big)^2k^2+\Big(\frac{1\cdot 3}{2\cdot 4}\Big)^2k^4 + \Big(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Big)^2k^6 + \ldots\ldots\Big],\ k^2<1.$$

INDEX.

[THE NUMERALS REFER TO PAGES.]